CHAPTER 1

Newton's Laws of Motion

1.1 Classical Mechanics

Mechanics is the study of how things move: how planets move around the sun, how a skier moves down the slope, or how an electron moves around the nucleus of an atom. So far as we know, the Greeks were the first to think seriously about mechanics, more than two thousand years ago, and the Greeks' mechanics represents a tremendous step in the evolution of modern science. Nevertheless, the Greek ideas were, by modern standards, seriously flawed and need not concern us here. The development of the mechanics that we know today began with the work of Galileo (1564–1642) and Newton (1642–1727), and it is the formulation of Newton, with his three laws of motion, that will be our starting point in this book.

In the late eighteenth and early nineteenth centuries, two alternative formulations of mechanics were developed, named for their inventors, the French mathematician and astronomer Lagrange (1736–1813) and the Irish mathematician Hamilton (1805–1865). The Lagrangian and Hamiltonian formulations of mechanics are completely equivalent to that of Newton, but they provide dramatically simpler solutions to many complicated problems and are also the taking-off point for various modern developments. The term *classical mechanics* is somewhat vague, but it is generally understood to mean these three equivalent formulations of mechanics, and it is in this sense that the subject of this book is called classical mechanics.

Until the beginning of the twentieth century, it seemed that classical mechanics was the *only* kind of mechanics, correctly describing all possible kinds of motion. Then, in the twenty years from 1905 to 1925, it became clear that classical mechanics did not correctly describe the motion of objects moving at speeds close to the speed of light, nor that of the microscopic particles inside atoms and molecules. The result was the development of two completely new forms of mechanics: relativistic mechanics to describe very high-speed motions and quantum mechanics to describe the motion of microscopic particles. I have included an introduction to relativity in the "optional" Chapter 15. Quantum mechanics requires a whole separate book (or several books), and I have made no attempt to give even a brief introduction to quantum mechanics.

Although classical mechanics has been replaced by relativistic mechanics and by quantum mechanics in their respective domains, there is still a vast range of interesting and topical problems in which classical mechanics gives a complete and accurate description of the possible motions. In fact, particularly with the advent of chaos theory in the last few decades, research in classical mechanics has intensified and the subject has become one of the most fashionable areas in physics. The purpose of this book is to give a thorough grounding in the exciting field of classical mechanics. When appropriate, I shall discuss problems in the framework of the Newtonian formulation, but I shall also try to emphasize those situations where the newer formulations of Lagrange and Hamilton are preferable and to use them when this is the case. At the level of this book, the Lagrangian approach has many significant advantages over the Newtonian, and we shall be using the Lagrangian formulation repeatedly, starting in Chapter 7. By contrast, the advantages of the Hamiltonian formulation show themselves only at a more advanced level, and I shall postpone the introduction of Hamiltonian mechanics to Chapter 13 (though it can be read at any point after Chapter 7).

In writing the book, I took for granted that you have had an introduction to Newtonian mechanics of the sort included in a typical freshman course in "General Physics." This chapter contains a brief review of the ideas that I assume you have met before.

1.2 Space and Time

Newton's three laws of motion are formulated in terms of four crucial underlying concepts: the notions of space, time, mass, and force. This section reviews the first two of these, space and time. In addition to a brief description of the classical view of space and time, I give a quick review of the machinery of vectors, with which we label the points of space.

Space

Each point P of the three-dimensional space in which we live can be labeled by a position vector $\mathbf{r}$ which specifies the distance and direction of P from a chosen origin O as in Figure 1.1. There are many different ways to identify a vector, of which one of the most natural is to give its components (x, y, z) in the directions of three chosen perpendicular axes. One popular way to express this is to introduce three unit vectors, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, pointing along the three axes and to write

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.1}$$

In elementary work, it is probably wise to choose a single good notation, such as (1.1), and stick with it. In more advanced work, however, it is almost impossible to avoid using several different notations. Different authors have different preferences (another popular choice is to use $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for what I am calling $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$) and you must get used to reading them all. Furthermore, almost every notation has its drawbacks, which can

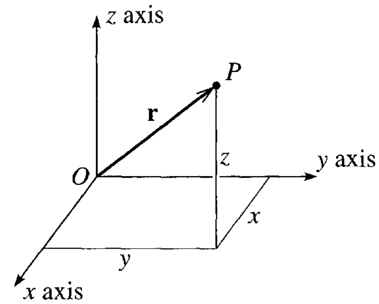

Figure 1.1 The point P is identified by its position vector $\mathbf{r}$, which gives the position of P relative to a chosen origin O. The vector $\mathbf{r}$ can be specified by its components (x, y, z) relative to chosen axes $Oxyz$.

make it unusable in some circumstances. Thus, while you may certainly choose your preferred scheme, you need to develop a tolerance for several different schemes.

It is sometimes convenient to be able to abbreviate (1.1) by writing simply

$$\mathbf{r} = (x, y, z). \tag{1.2}$$

This notation is obviously not quite consistent with (1.1), but it is usually completely unambiguous, asserting simply that $\mathbf{r}$ is the vector whose components are x, y, z. When the notation of (1.2) is the most convenient, I shall not hesitate to use it. For most vectors, we indicate the components by subscripts x, y, z. Thus the velocity vector $\mathbf{v}$ has components v_x, v_y, v_z and the acceleration $\mathbf{a}$ has components a_x, a_y, a_z.

As our equations become more complicated, it is sometimes inconvenient to write out all three terms in sums like (1.1); one would rather use the summation sign $\sum$ followed by a single term. The notation of (1.1) does not lend itself to this shorthand, and for this reason I shall sometimes relabel the three components x, y, z of $\mathbf{r}$ as r_1, r_2, r_3, and the three unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. That is, we define

$$r_1 = x, \qquad r_2 = y, \qquad r_3 = z,$$

and

$$\mathbf{e}_1 = \hat{\mathbf{x}}, \qquad \mathbf{e}_2 = \hat{\mathbf{y}}, \qquad \mathbf{e}_3 = \hat{\mathbf{z}}.$$

(The symbol $\mathbf{e}$ is commonly used for unit vectors, since $\mathbf{e}$ stands for the German "eins" or "one.") With these notations, (1.1) becomes

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = \sum_{i=1}^{3} r_i\mathbf{e}_i. \tag{1.3}$$

For a simple equation like this, the form (1.3) has no real advantage over (1.1), but with more complicated equations (1.3) is significantly more convenient, and I shall use this notation when appropriate.

Vector Operations

In our study of mechanics, we shall make repeated use of the various operations that can be performed with vectors. If $\mathbf{r}$ and $\mathbf{s}$ are vectors with components

$$\mathbf{r} = (r_1, r_2, r_3) \qquad \text{and} \qquad \mathbf{s} = (s_1, s_2, s_3),$$

then their **sum** (or resultant) $\mathbf{r} + \mathbf{s}$ is found by adding corresponding components, so that

$$\mathbf{r} + \mathbf{s} = (r_1 + s_1,\ r_2 + s_2,\ r_3 + s_3). \tag{1.4}$$

(You can convince yourself that this rule is equivalent to the familiar triangle and parallelogram rules for vector addition.) An important example of a vector sum is the resultant force on an object: When two forces $\mathbf{F}_a$ and $\mathbf{F}_b$ act on an object, the effect is the same as a single force, the resultant force, which is just the vector sum

$$\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b$$

as given by the vector addition law (1.4).

If c is a scalar (that is, an ordinary number) and $\mathbf{r}$ is a vector, the *product* $c\mathbf{r}$ is given by

$$c\mathbf{r} = (cr_1, cr_2, cr_3). \tag{1.5}$$

This means that $c\mathbf{r}$ is a vector in the same direction[1] as $\mathbf{r}$ with magnitude equal to c times the magnitude of $\mathbf{r}$. For example, if an object of mass m (a scalar) has an acceleration $\mathbf{a}$ (a vector), Newton's second law asserts that the resultant force $\mathbf{F}$ on the object will always equal the product $m\mathbf{a}$ as given by (1.5).

There are two important kinds of product that can be formed from any pair of vectors. First, the **scalar product** (or **dot product**) of two vectors $\mathbf{r}$ and $\mathbf{s}$ is given by either of the equivalent formulas

$$\mathbf{r} \cdot \mathbf{s} = rs\cos\theta \tag{1.6}$$

$$= r_1 s_1 + r_2 s_2 + r_3 s_3 = \sum_{n=1}^{3} r_n s_n \tag{1.7}$$

where r and s denote the magnitudes of the vectors $\mathbf{r}$ and $\mathbf{s}$, and θ is the angle between them. (For a proof that these two definitions are the same, see Problem 1.7.) For example, if a force $\mathbf{F}$ acts on an object that moves through a small displacement $d\mathbf{r}$, the work done by the force is the scalar product $\mathbf{F} \cdot d\mathbf{r}$, as given by either (1.6) or (1.7). Another important use of the scalar product is to define the magnitude of a vector: The magnitude (or length) of any vector $\mathbf{r}$ is denoted by $|\mathbf{r}|$ or r and, by Pythagoras's theorem is equal to $\sqrt{r_1^2 + r_2^2 + r_3^2}$. By (1.7) this is the same as

$$r = |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}}. \tag{1.8}$$

The scalar product $\mathbf{r} \cdot \mathbf{r}$ is often abbreviated as $\mathbf{r}^2$.

[1] Although this is what people usually say, one should actually be careful: If c is negative, $c\mathbf{r}$ is in the *opposite* direction to $\mathbf{r}$.

The second kind of product of two vectors $\mathbf{r}$ and $\mathbf{s}$ is the **vector product** (or **cross product**), which is defined as the vector $\mathbf{p} = \mathbf{r} \times \mathbf{s}$ with components

$$\left.\begin{aligned} p_x &= r_y s_z - r_z s_y \\ p_y &= r_z s_x - r_x s_z \\ p_z &= r_x s_y - r_y s_x \end{aligned}\right\} \tag{1.9}$$

or, equivalently

$$\mathbf{r} \times \mathbf{s} = \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix},$$

where "det" stands for the determinant. Either of these definitions implies that $\mathbf{r} \times \mathbf{s}$ is a vector perpendicular to both $\mathbf{r}$ and $\mathbf{s}$, with direction given by the familiar right-hand rule and magnitude $rs \sin\theta$ (Problem 1.15). The vector product plays an important role in the discussion of rotational motion. For example, the tendency of a force $\mathbf{F}$ (acting at a point $\mathbf{r}$) to cause a body to rotate about the origin is given by the torque of $\mathbf{F}$ about O, defined as the vector product $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$.

Differentiation of Vectors

Many (maybe most) of the laws of physics involve vectors, and most of these involve *derivatives* of vectors. There are so many ways to differentiate a vector that there is a whole subject called vector calculus, much of which we shall be developing in the course of this book. For now, I shall mention just the simplest kind of vector derivative, the time derivative of a vector that depends on time. For example, the velocity $\mathbf{v}(t)$ of a particle is the time derivative of the particle's position $\mathbf{r}(t)$; that is, $\mathbf{v} = d\mathbf{r}/dt$. Similarly the acceleration is the time derivative of the velocity, $\mathbf{a} = d\mathbf{v}/dt$.

The definition of the derivative of a vector is closely analogous to that of a scalar. Recall that if $x(t)$ is a scalar function of t, then we define its derivative as

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

where $\Delta x = x(t + \Delta t) - x(t)$ is the change in x as the time advances from t to $t + \Delta t$. In exactly the same way, if $\mathbf{r}(t)$ is any vector that depends on t, we define its derivative as

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \tag{1.10}$$

where

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \tag{1.11}$$

is the corresponding change in $\mathbf{r}(t)$. There are, of course, many delicate questions about the existence of this limit. Fortunately, none of these need concern us here: All of the vectors we shall encounter will be differentiable, and you can take for granted that the required limits exist. From the definition (1.10), one can prove that the derivative has all of the properties one would expect. For example, if $\mathbf{r}(t)$ and $\mathbf{s}(t)$

are two vectors that depend on t, then the derivative of their sum is just what you would expect:

$$\frac{d}{dt}(\mathbf{r}+\mathbf{s}) = \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{s}}{dt}. \tag{1.12}$$

Similarly, if $\mathbf{r}(t)$ is a vector and $f(t)$ is a scalar, then the derivative of the product $f(t)\mathbf{r}(t)$ is given by the appropriate version of the product rule,

$$\frac{d}{dt}(f\mathbf{r}) = f\frac{d\mathbf{r}}{dt} + \frac{df}{dt}\mathbf{r}. \tag{1.13}$$

If you are the sort of person who enjoys proving these kinds of proposition, you might want to show that they follow from the definition (1.10). Fortunately, if you do not enjoy this kind of activity, you don't need to worry, and you can safely take these results for granted.

One more result that deserves mention concerns the components of the derivative of a vector. Suppose that $\mathbf{r}$, with components x, y, z, is the position of a moving particle, and suppose that we want to know the particle's velocity $\mathbf{v} = d\mathbf{r}/dt$. When we differentiate the sum

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}, \tag{1.14}$$

the rule (1.12) gives us the sum of the three separate derivatives, and, by the product rule (1.13), each of these contains two terms. Thus, in principle, the derivative of (1.14) involves six terms in all. However, the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ do not depend on time, so their time derivatives are zero. Therefore, three of these six terms are zero, and we are left with just three terms:

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}. \tag{1.15}$$

Comparing this with the standard expansion

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$$

we see that

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad \text{and} \qquad v_z = \frac{dz}{dt}. \tag{1.16}$$

In words, the rectangular components of $\mathbf{v}$ are just the derivatives of the corresponding components of $\mathbf{r}$. This is a result that we use all the time (usually without even thinking about it) in solving elementary mechanics problems. What makes it especially noteworthy is this: It is true only because the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are constant, so that their derivatives are absent from (1.15). We shall find that in most coordinate systems, such as polar coordinates, the basic unit vectors are *not* constant, and the result corresponding to (1.16) is appreciably less transparent. In problems where we need to work in nonrectangular coordinates, it is considerably harder to write down velocities and accelerations in terms of the coordinates of $\mathbf{r}$, as we shall see.

Time

The classical view is that time is a single universal parameter t on which all observers agree. That is, if all observers are equipped with accurate clocks, all properly synchronized, then they will all agree as to the time at which any given event occurred. We know, of course, that this view is not exactly correct: According to the theory of relativity, two observers in relative motion do *not* agree on all times. Nevertheless, in the domain of classical mechanics, with all speeds much much less than the speed of light, the differences among the measured times are entirely negligible, and I shall adopt the classical assumption of a single universal time (except, of course, in Chapter 15 on relativity). Apart from the obvious ambiguity in the choice of the origin of time (the time that we choose to label $t = 0$), all observers agree on the times of all events.

Reference Frames

Almost every problem in classical mechanics involves a choice (explicit or implicit) of a *reference frame*, that is, a choice of spatial origin and axes to label positions as in Figure 1.1 and a choice of temporal origin to measure times. The difference between two frames may be quite minor. For instance, they may differ only in their choice of the origin of time — what one frame labels $t = 0$ the other may label $t' = t_o \neq 0$. Or the two frames may have the same origins of space and time, but have different orientations of the three spatial axes. By carefully choosing your reference frame, taking advantage of these different possibilities, you can sometimes simplify your work. For example, in problems involving blocks sliding down inclines, it often helps to choose one axis pointing down the slope.

A more important difference arises when two frames are in relative motion; that is, when one origin is moving relative to the other. In Section 1.4 we shall find that not all such frames are physically equivalent.[2] In certain special frames, called **inertial frames**, the basic laws hold true in their standard, simple form. (It is because one of these basic laws is Newton's first law, the law of inertia, that these frames are called inertial.) If a second frame is *accelerating or rotating* relative to an inertial frame, then this second frame is noninertial, and the basic laws — in particular, Newton's laws — do not hold in their standard form in this second frame. We shall find that the distinction between inertial and noninertial frames is central to our discussion of classical mechanics. It plays an even more explicit role in the theory of relativity.

1.3 Mass and Force

The concepts of mass and force are central to the formulation of classical mechanics. The proper definitions of these concepts have occupied many philosophers of science and are the subject of learned treatises. Fortunately we don't need to worry much about

[2] This statement is correct even in the theory of relativity.

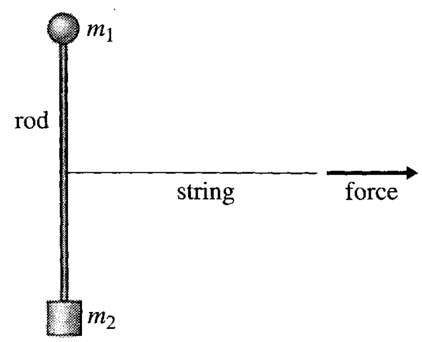

Figure 1.2 An inertial balance compares the masses m_1 and m_2 of two objects that are attached to the opposite ends of a rigid rod. The masses are equal if and only if a force applied at the rod's midpoint causes them to accelerate at the same rate, so that the rod does not rotate.

these delicate questions here. Based on your introductory course in general physics, you have a reasonably good idea what mass and force mean, and it is easy to describe how these parameters are defined and measured in many realistic situations.

Mass

The mass of an object characterizes the object's inertia — its resistance to being accelerated: A big boulder is hard to accelerate, and its mass is large. A little stone is easy to accelerate, and its mass is small. To make these natural ideas quantitative we have to define a unit of mass and then give a prescription for measuring the mass of any object in terms of the chosen unit. The internationally agreed unit of mass is the kilogram and is defined arbitrarily to be the mass of a chunk of platinum–iridium stored at the International Bureau of Weights and Measures outside Paris. To measure the mass of any other object, we need a means of comparing masses. In principle, this can be done with an inertial balance as shown in Figure 1.2. The two objects to be compared are fastened to the opposite ends of a light, rigid rod, which is then given a sharp pull at its midpoint. If the masses are equal, they will accelerate equally and the rod will move off without rotating; if the masses are unequal, the more massive one will accelerate less, and the rod will rotate as it moves off.

The beauty of the inertial balance is that it gives us a method of mass comparison that is based directly on the notion of mass as resistance to being accelerated. In practice, an inertial balance would be very awkward to use, and it is fortunate that there are much easier ways to compare masses, of which the easiest is to weigh the objects. As you certainly recall from your introductory physics course, an object's mass is found to be exactly proportional to the object's weight[3] (the gravitational force on the object) provided all measurements are made in the same location. Thus two

[3] This observation goes back to Galileo's famous experiments showing that all objects are accelerated at the same rate by gravity. The first modern experiments were conducted by the Hungarian physicist Eötvös (1848–1919), who showed that weight is proportional to mass to within

objects have the same mass if and only if they have the same weight (when weighed at the same place), and a simple, practical way to check whether two masses are equal is simply to weigh them and see if their weights are equal.

Armed with methods for comparing masses, we can easily set up a scheme to measure arbitrary masses. First, we can build a large number of standard kilograms, each one checked against the original 1-kg mass using either the inertial or gravitational balance. Next, we can build multiples and fractions of the kilogram, again checking them with our balance. (We check a 2-kg mass on one end of the balance against two 1-kg masses placed together on the other end; we check two half-kg masses by verifying that their masses are equal and that together they balance a 1-kg mass; and so on.) Finally, we can measure an unknown mass by putting it on one end of the balance and loading known masses on the other end until they balance to any desired precision.

Force

The informal notion of force as a push or pull is a surprisingly good starting point for our discussion of forces. We are certainly conscious of the forces that we exert ourselves. When I hold up a sack of cement, I am very aware that I am exerting an upward force on the sack; when I push a heavy crate across a rough floor, I am aware of the horizontal force that I have to exert in the direction of motion. Forces exerted by inanimate objects are a little harder to pin down, and we must, in fact, understand something of Newton's laws to identify such forces. If I let go of the sack of cement, it accelerates toward the ground; therefore, I conclude that there must be another force — the sack's weight, the gravitational force of the earth — pulling it downward. As I push the crate across the floor, I observe that it does not accelerate, and I conclude that there must be another force — friction — pushing the crate in the opposite direction. One of the most important skills for the student of elementary mechanics is to learn to examine an object's environment and identify all the forces on the object: What are the things touching the object and possibly exerting contact forces, such as friction or air pressure? And what are the nearby objects possibly exerting action-at-a-distance forces, such as the gravitational pull of the earth or the electrostatic force of some charged body?

If we accept that we know how to identify forces, it remains to decide how to measure them. As the unit of force we naturally adopt the newton (abbreviated N) defined as the magnitude of any single force that accelerates a standard kilogram mass with an acceleration of 1 m/s^2. Having agreed what we mean by one newton, we can proceed in several ways, all of which come to the same final conclusion, of course. The route that is probably preferred by most philosophers of science is to use Newton's second law to define the general force: A given force is 2 N if, by itself, it accelerates a standard kilogram with an acceleration of 2 m/s^2, and so

a few parts in 10^9. Experiments in the last few decades have narrowed this to around one part in 10^{12}.

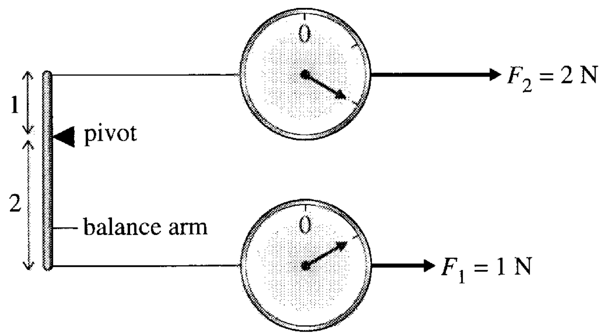

Figure 1.3 One of many possible ways to define forces of any magnitude. The lower spring balance has been calibrated to read 1 N. If the balance arm on the left is adjusted so that the lever arms above and below the pivot are in the ratio 1 : 2 and if the force F_1 is 1 N, then the force F_2 required to balance the arm is 2 N. This lets us calibrate the upper spring balance for 2 N. By readjusting the two lever arms, we can, in principle, calibrate the second spring balance to read any force.

on. This approach is not much like the way we usually measure forces in practice,[4] and for our present discussion a simpler procedure is to use some spring balances. Using our definition of the newton, we can calibrate a first spring balance to read 1 N. Then by matching a second spring balance against the first, using a balance arm as shown in Figure 1.3, we can define multiples and fractions of a newton. Once we have a fully calibrated spring balance we can, in principle, measure any unknown force, by matching it against the calibrated balance and reading off its value.

So far we have defined only the magnitude of a force. As you are certainly aware, forces are vectors, and we must also define their directions. This is easily done. If we apply a given force **F** (and no other forces) to any object at rest, the direction of **F** is defined as the direction of the resulting acceleration, that is, the direction in which the body moves off.

Now that we know, at least in principle, what we mean by positions, times, masses, and forces, we can proceed to discuss the cornerstone of our subject — Newton's three laws of motion.

[4] The approach also creates the confusing appearance that Newton's second law is just a consequence of the definition of force. This is not really true: Whatever definition we choose for force, a large part of the second law is experimental. One advantage of defining forces with spring balances is that it separates out the definition of force from the experimental basis of the second law. Of course, all commonly accepted definitions give the same final result for the value of any given force.

1.4 Newton's First and Second Laws; Inertial Frames

In this chapter, I am going to discuss Newton's laws as they apply to a **point mass**. A point mass, or **particle**, is a convenient fiction, an object with mass, but no size, that can move through space but has no internal degrees of freedom. It can have "translational" kinetic energy (energy of its motion through space) but no energy of rotation or of internal vibrations or deformations. Naturally, the laws of motion are simpler for point particles than for extended bodies, and this is the main reason that we start with the former. Later on, I shall build up the mechanics of extended bodies from our mechanics of point particles by considering the extended body as a collection of many separate particles.

Nevertheless, it is worth recognizing that there are many important problems where the objects of interest can be realistically approximated as point masses. Atomic and subatomic particles can often be considered to be point masses, and even macroscopic objects can frequently be approximated in this way. A stone thrown off the top of a cliff is, for almost all purposes, a point particle. Even a planet orbiting around the sun can usually be approximated in the same way. Thus the mechanics of point masses is more than just the starting point for the mechanics of extended bodies; it is a subject with wide application itself.

Newton's first two laws are well known and easily stated:

Newton's First Law (the Law of Inertia)

In the absence of forces, a particle moves with constant velocity $\mathbf{v}$.

and

Newton's Second Law

For any particle of mass m, the net force $\mathbf{F}$ on the particle is always equal to the mass m times the particle's acceleration:

$$\mathbf{F} = m\mathbf{a}. \tag{1.17}$$

In this equation $\mathbf{F}$ denotes the vector sum of all the forces on the particle and $\mathbf{a}$ is the particle's acceleration,

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} \\ &= \frac{d^2\mathbf{r}}{dt^2} \equiv \ddot{\mathbf{r}}.\end{aligned}$$

Here $\mathbf{v}$ denotes the particle's velocity, and I have introduced the convenient notation of dots to denote differentiation with respect to t, as in $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$.

Both laws can be stated in various equivalent ways. For instance (the first law): In the absence of forces, a stationary particle remains stationary and a moving particle continues to move with unchanging speed in the same direction. This is, of course, exactly the same as saying that the velocity is always constant. Again, $\mathbf{v}$ is constant if and only if the acceleration $\mathbf{a}$ is zero, so an even more compact statement is this: In the absence of forces a particle has zero acceleration.

The second law can be rephrased in terms of the particle's **momentum**, defined as

$$\mathbf{p} = m\mathbf{v}. \tag{1.18}$$

In classical mechanics, we take for granted that the mass m of a particle never changes, so that

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} = m\mathbf{a}.$$

Thus the second law (1.17) can be rephrased to say that

$$\mathbf{F} = \dot{\mathbf{p}}. \tag{1.19}$$

In classical mechanics, the two forms (1.17) and (1.19) of the second law are completely equivalent.[5]

Differential Equations

When written in the form $m\ddot{\mathbf{r}} = \mathbf{F}$, Newton's second law is a **differential equation** for the particle's position $\mathbf{r}(t)$. That is, it is an equation for the unknown function $\mathbf{r}(t)$ that involves *derivatives* of the unknown function. Almost all the laws of physics are, or can be cast as, differential equations, and a huge proportion of a physicist's time is spent solving these equations. In particular, most of the problems in this book involve differential equations — either Newton's second law or its counterparts in the Lagrangian and Hamiltonian forms of mechanics. These vary widely in their difficulty. Some are so easy to solve that one scarcely notices them. For example, consider Newton's second law for a particle confined to move along the x axis and subject to a constant force F_o,

$$\ddot{x}(t) = \frac{F_o}{m}.$$

This is a second-order differential equation for $x(t)$ as a function of t. (Second-order because it involves derivatives of second order, but none of higher order.) To solve it

[5] In relativity, the two forms are *not* equivalent, as we'll see in Chapter 15. Which form is correct depends on the definitions we use for force, mass, and momentum in relativity. If we adopt the most popular definitions of these three quantities, then it is the form (1.19) that holds in relativity.

one has only to integrate it twice. The first integration gives the velocity

$$\dot{x}(t) = \int \ddot{x}(t)\, dt = v_o + \frac{F_o}{m} t$$

where the constant of integration is the particle's initial velocity, and a second integration gives the position

$$x(t) = \int \dot{x}(t)\, dt = x_o + v_o t + \frac{F_o}{2m} t^2$$

where the second constant of integration is the particle's initial position. Solving this differential equation was so easy that we certainly needed no knowledge of the theory of differential equations. On the other hand, we shall meet lots of differential equations that do require knowledge of this theory, and I shall present the necessary theory as we need it. Obviously, it will be an advantage if you have already studied some of the theory of differential equations, but you should have no difficulty picking it up as we go along. Indeed, many of us find that the best way to learn this kind of mathematical theory is in the context of its physical applications.

Inertial Frames

On the face of it, Newton's second law includes his first: If there are no forces on an object, then $\mathbf{F} = 0$ and the second law (1.17) implies that $\mathbf{a} = 0$, which is the first law. There is, however, an important subtlety, and the first law has an important role to play. Newton's laws cannot be true in all conceivable reference frames. To see this, consider just the first law and imagine a reference frame — we'll call it $\mathcal{S}$ — in which the first law is true. For example, if the frame $\mathcal{S}$ has its origin and axes fixed relative to the earth's surface, then, to an excellent approximation, the first law (the law of inertia) holds with respect to the frame $\mathcal{S}$: A frictionless puck placed on a smooth horizontal surface is subject to zero force and, in accordance with the first law, it moves with constant velocity. Because the law of inertia holds, we call $\mathcal{S}$ an **inertial frame**. If we consider a second frame $\mathcal{S}'$ which is moving relative to $\mathcal{S}$ with constant velocity and is not rotating, then the same puck will also be observed to move with constant velocity relative to $\mathcal{S}'$. That is, the frame $\mathcal{S}'$ is also inertial.

If, however, we consider a third frame $\mathcal{S}''$ that is accelerating relative to $\mathcal{S}$, then, as viewed from $\mathcal{S}''$, the puck will be seen to be accelerating (in the opposite direction). Relative to the accelerating frame $\mathcal{S}''$ the law of inertia does not hold, and we say that $\mathcal{S}''$ is **noninertial**. I should emphasize that there is nothing mysterious about this result. Indeed it is a matter of experience. The frame $\mathcal{S}'$ could be a frame attached to a high-speed train traveling smoothly at constant speed along a straight track, and the frictionless puck, an ice cube placed on the floor of the train, as in Figure 1.4. As seen from the train (frame $\mathcal{S}'$), the ice cube is at rest and remains at rest, in accord with the first law. As seen from the ground (frame $\mathcal{S}$), the ice cube is moving with the same velocity as the train and continues to do so, again in obedience to the first law. But now consider conducting the same experiment on a second train (frame $\mathcal{S}''$) that is accelerating forward. As this train accelerates forward, the ice cube is left behind, and, relative to $\mathcal{S}''$, the ice cube accelerates backward, even though subject to no net

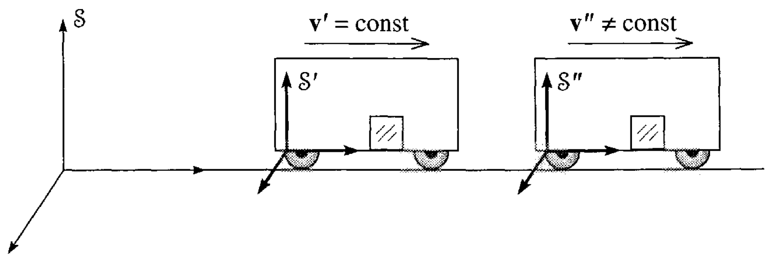

Figure 1.4 The frame $\mathcal{S}$ is fixed to the ground, while $\mathcal{S}'$ is fixed to a train traveling at constant velocity $\mathbf{v}'$ relative to $\mathcal{S}$. An ice cube placed on the floor of the train obeys Newton's first law as seen from both $\mathcal{S}$ and $\mathcal{S}'$. If the train to which $\mathcal{S}''$ is attached is accelerating forward, then, as seen in $\mathcal{S}''$, an ice cube placed on the floor will accelerate backward, and the first law does not hold in $\mathcal{S}''$.

force. Clearly the frame $\mathcal{S}''$ is noninertial, and neither of the first two laws can hold in $\mathcal{S}''$. A similar conclusion would hold if the frame $\mathcal{S}''$ had been attached to a rotating merry-go-round. A frictionless puck, subject to zero net force, would not move in a straight line as seen in $\mathcal{S}''$, and Newton's laws would not hold.

Evidently Newton's two laws hold only in the special, inertial (nonaccelerating and nonrotating) reference frames. Most philosophers of science take the view that the first law should be used to identify these inertial frames — a reference frame $\mathcal{S}$ is inertial if objects that are clearly subject to no forces are seen to move with constant velocity relative to $\mathcal{S}$.[6] Having identified the inertial frames by means of Newton's first law, we can then claim as an experimental fact that the second law holds in these same inertial frames.[7]

Since the laws of motion hold only in inertial frames, you might imagine that we would confine our attention exclusively to inertial frames, and, for a while, we shall do just that. Nevertheless, you should be aware that there are situations where it is necessary, or at least very convenient, to work in noninertial frames. The most important example of a noninertial frame is in fact the earth itself. To an excellent approximation, a reference frame fixed to the earth is inertial — a fortunate circumstance for students of physics! Nevertheless, the earth rotates on its axis once a day and circles around the sun once a year, and the sun orbits slowly around the center of the Milky Way galaxy. For all of these reasons, a reference frame fixed to the earth is not exactly inertial. Although these effects are very small, there are several phenomena — the tides and the trajectories of long-range projectiles are examples —

[6] There is some danger of going in a circle here: How do we know that the object is subject to no forces? We'd better not answer, "Because it's traveling at constant velocity"! Fortunately, we can argue that it is possible to identify all sources of force, such as people pushing and pulling or nearby massive bodies exerting gravitational forces. If there are no such things around, we can reasonably say that the object is free of forces.

[7] As I mentioned earlier, the extent to which the second law is an experimental statement depends on how we choose to define force. If we define force by means of the second law, then to some extent (though certainly not entirely) the law becomes a matter of definition. If we define forces by means of spring balances, then the second law is clearly an experimentally testable proposition.

that are most simply explained by taking into account the noninertial character of a frame fixed to the earth. In Chapter 9 we shall examine how the laws of motion must be modified for use in noninertial frames. For the moment, however, we shall confine our discussion to inertial frames.

Validity of the First Two Laws

Since the advent of relativity and quantum mechanics, we have known that Newton's laws are not universally valid. Nevertheless, there is an immense range of phenomena — the phenomena of classical physics — where the first two laws are for all practical purposes exact. Even as the speeds of interest approach c, the speed of light, and relativity becomes important, the first law remains exactly true. (In relativity, just as in classical mechanics, an inertial frame is *defined* as one where the first law holds.)[8] As we shall see in Chapter 15, the two forms of the second law, $\mathbf{F} = m\mathbf{a}$ and $\mathbf{F} = \dot{\mathbf{p}}$, are no longer equivalent in relativity, although with $\mathbf{F}$ and $\mathbf{p}$ suitably defined the second law in the form $\mathbf{F} = \dot{\mathbf{p}}$ is still valid. In any case, the important point is this: In the classical domain, we can and shall assume that the first two laws (the second in either form) are universally and precisely valid. You can, if you wish, regard this assumption as defining a model — the classical model — of the natural world. The model is logically consistent and is such a good representation of many phenomena that it is amply worthy of our study.

1.5 The Third Law and Conservation of Momentum

Newton's first two laws concern the response of a single object to applied forces. The third law addresses a quite different issue: Every force on an object inevitably involves a second object — the object *that exerts the force*. The nail is hit *by the hammer*, the cart is pulled *by the horse*, and so on. While this much is no doubt a matter of common sense, the third law goes considerably beyond our everyday experience. Newton realized that if an object 1 exerts a force on another object 2, then object 2 always exerts a force (the "reaction" force) back on object 1. This seems quite natural: If you push hard against a wall, it is fairly easy to convince yourself that the wall is exerting a force back on you, without which you would undoubtedly fall over. The aspect of the third law which certainly goes beyond our normal perceptions is this: According to the third law, the reaction force of object 2 on object 1 is always equal and opposite to the original force of 1 on 2. If we introduce the notation $\mathbf{F}_{21}$ to denote the force exerted on object 2 by object 1, Newton's third law can be stated very compactly:

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[8] However, in relativity the relationship between different inertial frames — the so-called Lorentz transformation — is different from that of classical mechanics. See Section 15.6.

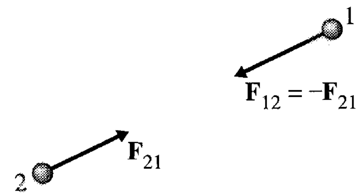

Figure 1.5 Newton's third law asserts that the reaction force exerted on object 1 by object 2 is equal and opposite to the force exerted on 2 by 1, that is, $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Newton's Third Law

If object 1 exerts a force $\mathbf{F}_{21}$ on object 2, then object 2 always exerts a reaction force $\mathbf{F}_{12}$ on object 1 given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{1.20}$$

This statement is illustrated in Figure 1.5, which you could think of as showing the force of the earth on the moon and the reaction force of the moon on the earth (or a proton on an electron and the electron on the proton). Notice that this figure actually goes a little beyond the usual statement (1.20) of the third law: Not only have I shown the two forces as equal and opposite; I have also shown them acting along the line joining 1 and 2. Forces with this extra property are called **central forces**. (They act along the line of centers.) The third law does not actually require that the forces be central, but, as I shall discuss later, most of the forces we encounter (gravity, the electrostatic force between two charges, etc.) do have this property.

As Newton himself was well aware, the third law is intimately related to the law of conservation of momentum. Let us focus, at first, on just two objects as shown in Figure 1.6, which might show the earth and the moon or two skaters on the ice. In addition to the force of each object on the other, there may be "external" forces exerted by other bodies. The earth and moon both experience forces exerted by the sun, and both skaters could experience the external force of the wind. I have shown the net external forces on the two objects as $\mathbf{F}_1^{\text{ext}}$ and $\mathbf{F}_2^{\text{ext}}$. The total force on object 1 is then

$$(\text{net force on 1}) \equiv \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_1^{\text{ext}}$$

and similarly

$$(\text{net force on 2}) \equiv \mathbf{F}_2 = \mathbf{F}_{21} + \mathbf{F}_2^{\text{ext}}.$$

We can compute the rates of change of the particles' momenta using Newton's second law:

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_1^{\text{ext}} \tag{1.21}$$

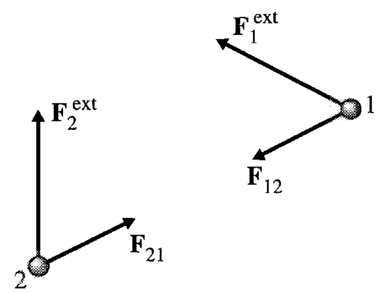

Figure 1.6 Two objects exert forces on each other and may also be subject to additional "external" forces from other objects not shown.

and

$$\dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_{21} + \mathbf{F}_2^{\text{ext}}. \tag{1.22}$$

If we now define the total momentum of our two objects as

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,$$

then the rate of change of the total momentum is just

$$\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2.$$

To evaluate this, we have only to add Equations (1.21) and (1.22). When we do this, the two internal forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, cancel out because of Newton's third law, and we are left with

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} \equiv \mathbf{F}^{\text{ext}}, \tag{1.23}$$

where I have introduced the notation $\mathbf{F}^{\text{ext}}$ to denote the total external force on our two-particle system.

The result (1.23) is the first in a series of important results that let us construct a theory of many-particle systems from the basic laws for a single particle. It asserts that as far as the total momentum of a system is concerned, the internal forces have no effect. A special case of this result is that if there are no external forces ($\mathbf{F}^{\text{ext}} = 0$) then $\dot{\mathbf{P}} = 0$. Thus we have the important result:

$$\text{If} \qquad \mathbf{F}^{\text{ext}} = 0, \qquad \text{then} \qquad \mathbf{P} = \text{const.} \tag{1.24}$$

In the absence of external forces, the total momentum of our two-particle system is constant — a result called the principle of conservation of momentum.

Multiparticle Systems

We have proved the conservation of momentum, Equation (1.24), for a system of two particles. The extension of the result to any number of particles is straightforward in principle, but I would like to go through it in detail, because it lets me introduce some

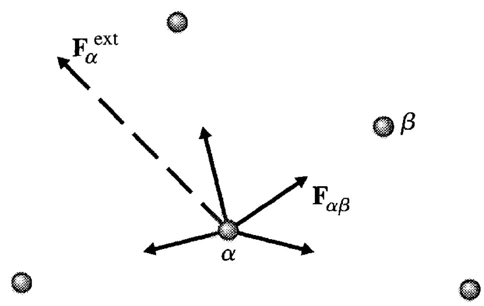

Figure 1.7 A five-particle system with particles labelled by α or $\beta = 1, 2, \cdots, 5$. The particle α is subject to four internal forces, shown by solid arrows and denoted $\mathbf{F}_{\alpha\beta}$ (the force on α by β). In addition particle α may be subject to a net external force, shown by the dashed arrow and denoted $\mathbf{F}_\alpha^{\text{ext}}$.

important notation and will give you some practice using the summation notation. Let us consider then a system of N particles. I shall label the typical particle with a Greek index α or β, either of which can take any of the values $1, 2, \cdots, N$. The mass of particle α is m_α and its momentum is $\mathbf{p}_\alpha$. The force on particle α is quite complicated: Each of the other $(N - 1)$ particles can exert a force which I shall call $\mathbf{F}_{\alpha\beta}$, the force on α by β, as illustrated in Figure 1.7. In addition there may be a net external force on particle α, which I shall call $\mathbf{F}_\alpha^{\text{ext}}$. Thus the net force on particle α is

$$(\text{net force on particle } \alpha) = \mathbf{F}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}}. \tag{1.25}$$

Here the sum runs over all values of β not equal to α. (Remember there is no force $\mathbf{F}_{\alpha\alpha}$ because particle α cannot exert a force on itself.) According to Newton's second law, this is the same as the rate of change of $\mathbf{p}_\alpha$:

$$\dot{\mathbf{p}}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}}. \tag{1.26}$$

This result holds for each $\alpha = 1, \cdots, N$.

Let us now consider the total momentum of our N-particle system,

$$\mathbf{P} = \sum_\alpha \mathbf{p}_\alpha$$

where, of course, this sum runs over all N particles, $\alpha = 1, 2, \cdots, N$. If we differentiate this equation with respect to time, we find

$$\dot{\mathbf{P}} = \sum_\alpha \dot{\mathbf{p}}_\alpha$$

or, substituting for $\dot{\mathbf{p}}_\alpha$ from (1.26),

$$\dot{\mathbf{P}} = \sum_\alpha \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \sum_\alpha \mathbf{F}_\alpha^{\text{ext}}. \tag{1.27}$$

The double sum here contains $N(N-1)$ terms in all. Each term $\mathbf{F}_{\alpha\beta}$ in this sum can be paired with a second term $\mathbf{F}_{\beta\alpha}$ (that is, $\mathbf{F}_{12}$ paired with $\mathbf{F}_{21}$, and so on), so that

$$\sum_{\alpha}\sum_{\beta\neq\alpha}\mathbf{F}_{\alpha\beta}=\sum_{\alpha}\sum_{\beta>\alpha}(\mathbf{F}_{\alpha\beta}+\mathbf{F}_{\beta\alpha}). \tag{1.28}$$

The double sum on the right includes only values of α and β with $\alpha<\beta$ and has half as many terms as that on the left. But each term is the sum of two forces, $(\mathbf{F}_{\alpha\beta}+\mathbf{F}_{\beta\alpha})$, and, by the third law, each such sum is zero. Therefore the whole double sum in (1.28) is zero, and returning to (1.27) we conclude that

$$\dot{\mathbf{P}}=\sum_{\alpha}\mathbf{F}_{\alpha}^{\text{ext}}\equiv\mathbf{F}^{\text{ext}}. \tag{1.29}$$

The result (1.29) corresponds exactly to the two-particle result (1.23). Like the latter, it says that the internal forces have no effect on the evolution of the total momentum $\mathbf{P}$ — the rate of change of $\mathbf{P}$ is determined by the net *external* force on the system. In particular, if the net external force is zero, we have the

Principle of Conservation of Momentum

If the net external force $\mathbf{F}^{\text{ext}}$ on an N-particle system is zero, the system's total momentum $\mathbf{P}$ is constant.

As you are certainly aware, this is one of the most important results in classical physics and is, in fact, also true in relativity and quantum mechanics. If you are not very familiar with the sorts of manipulations of sums that we used, it would be a good idea to go over the argument leading from (1.25) to (1.29) for the case of three or four particles, writing out all the sums explicitly (Problems 1.28 or 1.29). You should also convince yourself that, conversely, if the principle of conservation of momentum is true for all multiparticle systems, then Newton's third law must be true (Problem 1.31). In other words, conservation of momentum and Newton's third law are equivalent to one another.

Validity of Newton's Third Law

Within the domain of classical physics, the third law, like the second, is valid with such accuracy that it can be taken to be exact. As speeds approach the speed of light, it is easy to see that the third law cannot hold: The point is that the law asserts that the action and reaction forces, $\mathbf{F}_{12}(t)$ and $\mathbf{F}_{21}(t)$, *measured at the same time* t, are equal and opposite. As you certainly know, once relativity becomes important the concept of a single universal time has to be abandoned — two events that are seen as simultaneous by one observer are, in general, *not* simultaneous as seen by a second observer. Thus, even if the equality $\mathbf{F}_{12}(t)=-\mathbf{F}_{21}(t)$ (with both times the same) were true for one observer, it would generally be false for another. Therefore, the third law cannot be valid once relativity becomes important.

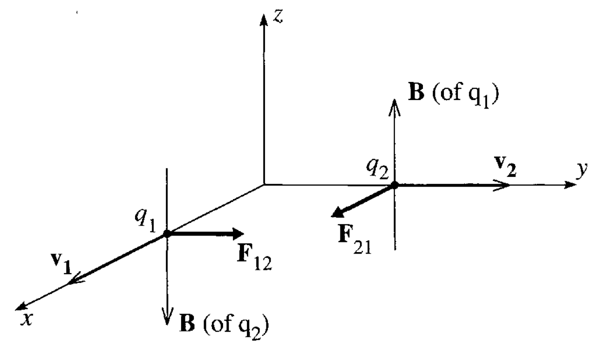

Figure 1.8 Each of the positive charges q_1 and q_2 produces a magnetic field that exerts a force on the other charge. The resulting magnetic forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ do not obey Newton's third law.

Rather surprisingly, there is a simple example of a well-known force — the magnetic force between two moving charges — for which the third law is not exactly true, even at slow speeds. To see this, consider the two positive charges of Figure 1.8, with q_1 moving in the x direction and q_2 moving in the y direction, as shown. The exact calculation of the magnetic field produced by each charge is complicated, but a simple argument gives the correct directions of the two fields, and this is all we need. The moving charge q_1 is equivalent to a current in the x direction. By the right-hand rule for fields, this produces a magnetic field which points in the z direction in the vicinity of q_2. By the right-hand rule for forces, this field produces a force $\mathbf{F}_{21}$ on q_2 that is in the x direction. An exactly analogous argument (check it yourself) shows that the force $\mathbf{F}_{12}$ on q_1 is in the y direction, as shown. Clearly these two forces do not obey Newton's third law!

This conclusion is especially startling since we have just seen that Newton's third law is equivalent to the conservation of momentum. Apparently the total momentum $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ of the two charges in Figure 1.8 is not conserved. This conclusion, which is correct, serves to remind us that the "mechanical" momentum $m\mathbf{v}$ of particles is not the only kind of momentum. Electromagnetic fields can also carry momentum, and in the situation of Figure 1.8 the mechanical momentum being lost by the two particles is going to the electromagnetic momentum of the fields.

Fortunately, if both speeds in Figure 1.8 are much less than the speed of light ($v \ll c$), the loss of mechanical momentum and the concomitant failure of the third law are completely negligible. To see this, note that in addition to the magnetic force between q_1 and q_2 there is the electrostatic Coulomb force[9] kq_1q_2/r^2, which *does* obey Newton's third law. It is a straightforward exercise (Problem 1.32) to show that the magnetic force is of order v^2/c^2 times the Coulomb force. Thus only as v approaches c — and classical mechanics must give way to relativity anyway — is the violation of

[9] Here k is the Coulomb force constant, often written as $k = 1/(4\pi\epsilon_o)$.

the third law by the magnetic force important.[10] We see that the unexpected situation of Figure 1.8 does not contradict our claim that in the classical domain Newton's third law is valid, and this is what we shall assume in our discussions of nonrelativistic mechanics.

1.6 Newton's Second Law in Cartesian Coordinates

Of Newton's three laws, the one that we actually use the most is the second, which is often described as the *equation of motion*. As we have seen, the first is theoretically important to define what we mean by inertial frames but is usually of no practical use beyond this. The third law is crucially important in sorting out the internal forces in a multiparticle system, but, once we know the forces involved, the second law is what we actually use to calculate the motion of the object or objects of interest. In particular, in many simple problems the forces are known or easily found, and, in this case, the second law is all we need for solving the problem.

As we have already noted, the second law,

$$\mathbf{F} = m\ddot{\mathbf{r}}, \tag{1.30}$$

is a second-order, differential equation[11] for the position vector $\mathbf{r}$ as a function of the time t. In the prototypical problem, the forces that comprise $\mathbf{F}$ are given, and our job is to solve the differential equation (1.30) for $\mathbf{r}(t)$. Sometimes we are told about $\mathbf{r}(t)$, and we have to use (1.30) to find some of the forces. In any case, the equation (1.30) is a *vector* differential equation. And the simplest way to solve such equations is almost always to resolve the vectors into their components relative to a chosen coordinate system.

Conceptually the simplest coordinate system is the Cartesian (or rectangular), with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, in terms of which the net force $\mathbf{F}$ can then be written as

$$\mathbf{F} = F_x\,\hat{\mathbf{x}} + F_y\,\hat{\mathbf{y}} + F_z\,\hat{\mathbf{z}} \tag{1.31}$$

and the position vector $\mathbf{r}$ as

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}. \tag{1.32}$$

As we noted in Section 1.2, this expansion of $\mathbf{r}$ in terms of its Cartesian components is especially easy to differentiate because the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are constant. Thus we can differentiate (1.32) twice to get the simple result

$$\ddot{\mathbf{r}} = \ddot{x}\,\hat{\mathbf{x}} + \ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}. \tag{1.33}$$

[10] The magnetic force between two steady currents is not necessarily small, even in the classical domain, but it can be shown that this force *does* obey the third law. See Problem 1.33.

[11] The force $\mathbf{F}$ can sometimes involve derivatives of $\mathbf{r}$. (For instance the magnetic force on a moving charge involves the velocity $\mathbf{v} = \dot{\mathbf{r}}$.) Very occasionally the force $\mathbf{F}$ involves a higher derivative of $\mathbf{r}$, of order $n > 2$, in which case the second law is an nth-order differential equation.

That is, the three Cartesian components of $\ddot{\mathbf{r}}$ are just the appropriate derivatives of the three coordinates x, y, z of $\mathbf{r}$, and the second law (1.30) becomes

$$F_x\,\hat{\mathbf{x}} + F_y\,\hat{\mathbf{y}} + F_z\,\hat{\mathbf{z}} = m\ddot{x}\,\hat{\mathbf{x}} + m\ddot{y}\,\hat{\mathbf{y}} + m\ddot{z}\,\hat{\mathbf{z}}. \tag{1.34}$$

Resolving this equation into its three separate components, we see that F_x has to equal $m\ddot{x}$ and similarly for the y and z components. That is, in Cartesian coordinates, the single vector equation (1.30) is equivalent to the three separate equations:

$$\mathbf{F} = m\ddot{\mathbf{r}} \qquad \Longleftrightarrow \qquad \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z}. \end{cases} \tag{1.35}$$

This beautiful result, that, in Cartesian coordinates, Newton's second law in three dimensions is equivalent to three one-dimensional versions of the same law, is the basis of the solution of almost all simple mechanics problems in Cartesian coordinates. Here is an example to remind you of how such problems go.

EXAMPLE 1.1 A Block Sliding down an Incline

A block of mass m is observed accelerating from rest down an incline that has coefficient of friction μ and is at angle θ from the horizontal. How far will it travel in time t?

Our first task is to choose our frame of reference. Naturally, we choose our spatial origin at the block's starting position and the origin of time ($t = 0$) at the moment of release. As you no doubt remember from your introductory physics course, the best choice of axes is to have one axis (x say) point down the slope, one (y) normal to the slope, and the third (z) across it, as shown in Figure 1.9. This choice has two advantages: First, because the block slides straight down the slope, the motion is entirely in the x direction, and only x varies. (If we had chosen the x axis horizontal and the y axis vertical, then both x and y would vary.) Second, two of the three forces on the block are unknown (the normal force $\mathbf{N}$ and friction $\mathbf{f}$; the weight, $\mathbf{w} = m\mathbf{g}$, we treat as known), and with our choice of axes, each of the unknowns has only one nonzero component, since $\mathbf{N}$ is in the y direction and $\mathbf{f}$ is in the (negative) x direction.

We are now ready to apply Newton's second law. The result (1.35) means that we can analyse the three components separately, as follows:

There are no forces in the z direction, so $F_z = 0$. Since $F_z = m\ddot{z}$, it follows that $\ddot{z} = 0$, which implies that $\dot{z}$ (or v_z) is constant. Since the block starts from rest, this means that $\dot{z}$ is actually zero for all t. With $\dot{z} = 0$, it follows that z is constant, and, since it too starts from zero, we conclude that $z = 0$ for all t. As we would certainly have guessed, the motion remains in the xy plane.

Since the block does not jump off the incline, we know that there is no motion in the y direction. In particular, $\ddot{y} = 0$. Therefore, Newton's second law implies that the y component of the net force is zero; that is, $F_y = 0$. From Figure 1.9 we see that this implies that

$$F_y = N - mg\cos\theta = 0.$$

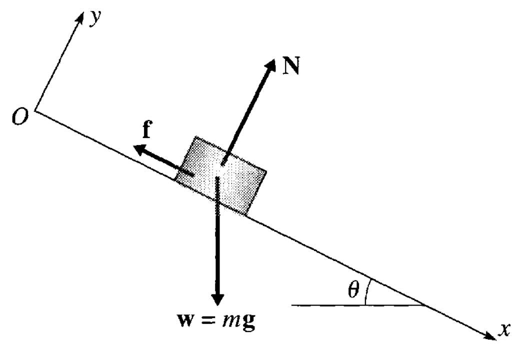

Figure 1.9 A block slides down a slope of incline θ. The three forces on the block are its weight, $\mathbf{w} = m\mathbf{g}$, the normal force of the incline, $\mathbf{N}$, and the frictional force $\mathbf{f}$, whose magnitude is $f = \mu N$. The z axis is not shown but points out of the page, that is, across the slope.

Thus the y component of the second law has told us that the unknown normal force is $N = mg\cos\theta$. Since $f = \mu N$, this tells us the frictional force, $f = \mu mg\cos\theta$, and all the forces are now known. All that remains is to use the remaining component (the x component) of the second law to solve for the actual motion.

The x component of the second law, $F_x = m\ddot{x}$, implies (see Figure 1.9) that

$$w_x - f = m\ddot{x}$$

or

$$mg\sin\theta - \mu mg\cos\theta = m\ddot{x}.$$

The m's cancel, and we find for the acceleration down the slope

$$\ddot{x} = g(\sin\theta - \mu\cos\theta). \tag{1.36}$$

Having found $\ddot{x}$, and found it to be constant, we have only to integrate it twice to find x as a function of t. First

$$\dot{x} = g(\sin\theta - \mu\cos\theta)t.$$

(Remember that $\dot{x} = 0$ initially, so the constant of integration is zero.) Finally,

$$x(t) = \tfrac{1}{2}g(\sin\theta - \mu\cos\theta)t^2$$

(again the constant of integration is zero) and our solution is complete.

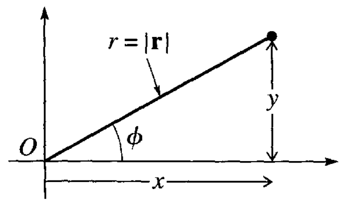

Figure 1.10 The definition of the polar coordinates r and ϕ.

1.7 Two-Dimensional Polar Coordinates

While Cartesian coordinates have the merit of simplicity, we are going to find that it is almost impossible to solve certain problems without the use of various non-Cartesian coordinate systems. To illustrate the complexities of non-Cartesian coordinates, let us consider the form of Newton's second law in a two-dimensional problem using polar coordinates. These coordinates are defined in Figure 1.10. Instead of using the two rectangular coordinates x, y, we label the position of a particle with its distance r from O and the angle ϕ measured up from the x axis. Given the rectangular coordinates x and y, you can calculate the polar coordinates r and ϕ, or vice versa, using the following relations. (Make sure you understand all four equations.[12])

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \phi &= \arctan(y/x) \end{aligned}\right. \tag{1.37}$$

Just as with rectangular coordinates, it is convenient to introduce two unit vectors, which I shall denote by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$. To understand their definitions, notice that we can define the unit vector $\hat{\mathbf{x}}$ as the unit vector that points in the direction of increasing x when y is fixed, as shown in Figure 1.11(a). In the same way we shall define $\hat{\mathbf{r}}$ as the unit vector that points in the direction we move when r increases with ϕ fixed; likewise, $\hat{\boldsymbol{\phi}}$ is the unit vector that points in the direction we move when ϕ increases with r fixed. Figure 1.11 makes clear a most important difference between the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ of rectangular coordinates and our new unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$. The vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the same at all points in the plane, whereas the new vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ change their directions as the position vector $\mathbf{r}$ moves around. We shall see that this complicates the use of Newton's second law in polar coordinates.

Figure 1.11 suggests another way to write the unit vector $\hat{\mathbf{r}}$. Since $\hat{\mathbf{r}}$ is in the same direction as $\mathbf{r}$, but has magnitude 1, you can see that

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}. \tag{1.38}$$

This result suggests a second role for the "hat" notation. For *any* vector $\mathbf{a}$, we can define $\hat{\mathbf{a}}$ as the unit vector in the direction of $\mathbf{a}$, namely $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$.

[12] There is a small subtlety concerning the equation for ϕ: You need to make sure ϕ lands in the proper quadrant, since the first and third quadrants give the same values for y/x (and likewise the second and fourth). See Problem 1.42.

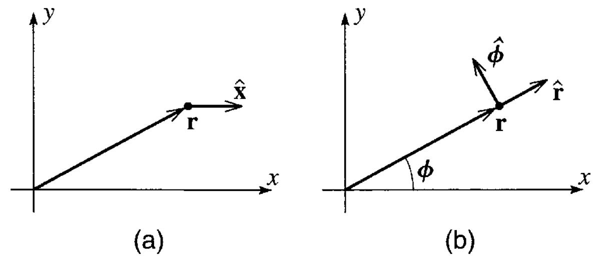

Figure 1.11 **(a)** The unit vector $\hat{\mathbf{x}}$ points in the direction of increasing x with y fixed. **(b)** The unit vector $\hat{\mathbf{r}}$ points in the direction of increasing r with ϕ fixed; $\hat{\boldsymbol{\phi}}$ points in the direction of increasing ϕ with r fixed. Unlike $\hat{\mathbf{x}}$, the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ change as the position vector $\mathbf{r}$ moves.

Since the two unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are perpendicular vectors in our two-dimensional space, any vector can be expanded in terms of them. For instance, the net force $\mathbf{F}$ on an object can be written

$$\mathbf{F} = F_r\hat{\mathbf{r}} + F_\phi\hat{\boldsymbol{\phi}}. \tag{1.39}$$

If, for example, the object in question is a stone that I am twirling in a circle on the end of a string (with my hand at the origin), then F_r would be the tension in the string and F_ϕ the force of air resistance retarding the stone in the tangential direction. The expansion of the position vector itself is especially simple in polar coordinates. From Figure 1.11(b) it is clear that

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{1.40}$$

We are now ready to ask about the form of Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, in polar coordinates. In rectangular coordinates, we saw that the x component of $\ddot{\mathbf{r}}$ is just $\ddot{x}$, and this is what led to the very simple result (1.35). We must now find the components of $\ddot{\mathbf{r}}$ in polar coordinates; that is, we must differentiate (1.40) with respect to t. Although (1.40) is very simple, the vector $\hat{\mathbf{r}}$ changes as $\mathbf{r}$ moves. Thus when we differentiate (1.40), we shall pick up a term involving the derivative of $\hat{\mathbf{r}}$. Our first task is to find this derivative of $\hat{\mathbf{r}}$.

Figure 1.12(a) shows the position of the particle of interest at two successive times, t_1 and $t_2 = t_1 + \Delta t$. If the corresponding angles $\phi(t_1)$ and $\phi(t_2)$ are different, then the two unit vectors $\hat{\mathbf{r}}(t_1)$ and $\hat{\mathbf{r}}(t_2)$ point in different directions. The change in $\hat{\mathbf{r}}$ is shown in Figure 1.12(b), and (provided Δt is small) is approximately

$$\begin{aligned}\Delta\hat{\mathbf{r}} &\approx \Delta\phi\,\hat{\boldsymbol{\phi}}\\ &\approx \dot{\phi}\,\Delta t\,\hat{\boldsymbol{\phi}}.\end{aligned} \tag{1.41}$$

(Notice that the direction of $\Delta\hat{\mathbf{r}}$ is perpendicular to $\hat{\mathbf{r}}$, namely the direction of $\hat{\boldsymbol{\phi}}$.) If we divide both sides by Δt and take the limit as $\Delta t \to 0$, then $\Delta\hat{\mathbf{r}}/\Delta t \to d\hat{\mathbf{r}}/dt$ and we find that

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{1.42}$$

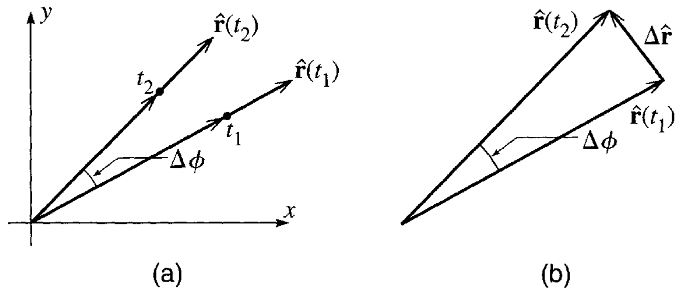

Figure 1.12 **(a)** The positions of a particle at two successive times, t_1 and t_2. Unless the particle is moving exactly radially, the corresponding unit vectors $\hat{\mathbf{r}}(t_1)$ and $\hat{\mathbf{r}}(t_2)$ point in different directions. **(b)** The change $\Delta\hat{\mathbf{r}}$ in $\hat{\mathbf{r}}$ is given by the triangle shown.

(For an alternative proof of this important result, see Problem 1.43.) Notice that $d\hat{\mathbf{r}}/dt$ is in the direction of $\hat{\boldsymbol{\phi}}$ and is proportional to the rate of change of the angle ϕ — both of which properties we would expect based on Figure 1.12.

Now that we know the derivative of $\hat{\mathbf{r}}$, we are ready to differentiate Equation (1.40). Using the product rule, we get two terms:

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt},$$

and, substituting (1.42), we find for the velocity $\dot{\mathbf{r}}$, or $\mathbf{v}$,

$$\mathbf{v} \equiv \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{1.43}$$

From this we can read off the polar components of the velocity:

$$v_r = \dot{r} \quad \text{and} \quad v_\phi = r\dot{\phi} = r\omega \tag{1.44}$$

where in the second equation I have introduced the traditional notation ω for the angular velocity $\dot{\phi}$. While the results in (1.44) should be familiar from your introductory physics course, they are undeniably more complicated than the corresponding results in Cartesian coordinates ($v_x = \dot{x}$ and $v_y = \dot{y}$).

Before we can write down Newton's second law, we have to differentiate a second time to find the acceleration:

$$\mathbf{a} \equiv \ddot{\mathbf{r}} = \frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}), \tag{1.45}$$

where the final expression comes from substituting (1.43) for $\dot{\mathbf{r}}$. To complete the differentiation in (1.45), we must calculate the derivative of $\hat{\boldsymbol{\phi}}$. This calculation is completely analogous to the argument leading to (1.42) and is illustrated in Figure 1.13. By inspecting this figure, you should be able to convince yourself that

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\mathbf{r}}. \tag{1.46}$$

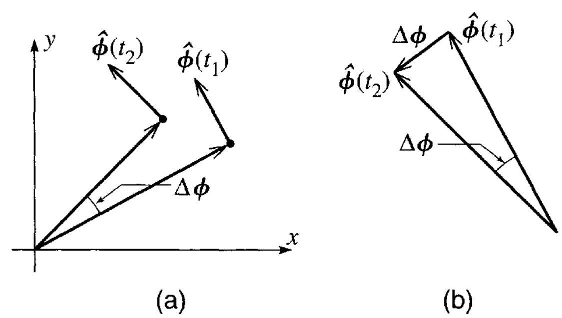

Figure 1.13 **(a)** The unit vector $\hat{\boldsymbol{\phi}}$ at two successive times t_1 and t_2. **(b)** The change $\Delta\hat{\boldsymbol{\phi}}$.

Returning to Equation (1.45), we can now carry out the differentiation to give the following five terms:

$$\mathbf{a} = \left(\ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt}\right) + \left((\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt}\right)$$

or, if we use (1.42) and (1.46) to replace the derivatives of the two unit vectors,

$$\mathbf{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}}. \tag{1.47}$$

This horrible result is a little easier to understand if we consider the special case that r is constant, as is the case for a stone that I twirl on the end of a string of fixed length. With r constant, both derivatives of r are zero, and (1.47) has just two terms:

$$\mathbf{a} = -r\dot{\phi}^2\hat{\mathbf{r}} + r\ddot{\phi}\hat{\boldsymbol{\phi}}$$

or

$$\mathbf{a} = -r\omega^2\hat{\mathbf{r}} + r\alpha\hat{\boldsymbol{\phi}},$$

where $\omega = \dot{\phi}$ denotes the angular velocity and $\alpha = \ddot{\phi}$ is the angular acceleration. This is the familiar result from elementary physics that when a particle moves around a fixed circle, it has an inward "centripetal" acceleration $r\omega^2$ (or v^2/r) and a tangential acceleration, $r\alpha$. Nevertheless, when r is not constant, the acceleration includes all four of the terms in (1.47). The first term, $\ddot{r}$ in the radial direction is what you would probably expect when r varies, but the final term, $2\dot{r}\dot{\phi}$ in the ϕ direction, is harder to understand. It is called the Coriolis acceleration, and I shall discuss it in detail in Chapter 9.

Having calculated the acceleration as in (1.47), we can finally write down Newton's second law in terms of polar coordinates:

$$\mathbf{F} = m\mathbf{a} \qquad \Longleftrightarrow \qquad \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}). \end{cases} \tag{1.48}$$

These equations in polar coordinates are a far cry from the beautifully simple equations (1.35) for rectangular coordinates. In fact, one of the main reasons for taking the trouble to recast Newtonian mechanics in the Lagrangian formulation (Chapter 7) is that the latter is able to handle nonrectangular coordinates just as easily as rectangular.

You may justifiably be feeling that the second law in polar coordinates is so complicated that there could be no occasion to use it. In fact, however, there are many problems which are most easily solved using polar coordinates, and I conclude this section with an elementary example.

EXAMPLE 1.2 An Oscillating Skateboard

A "half-pipe" at a skateboard park consists of a concrete trough with a semicircular cross section of radius $R = 5$ m, as shown in Figure 1.14. I hold a frictionless skateboard on the side of the trough pointing down toward the bottom and release it. Discuss the subsequent motion using Newton's second law. In particular, if I release the board just a short way from the bottom, how long will it take to come back to the point of release?

Because the skateboard is constrained to move on a circular path, this problem is most easily solved using polar coordinates with origin O at the center of the pipe as shown. (At some point in the following calculation, try writing the second law in rectangular coordinates and observe what a tangle you get.) With this choice of polar coordinates, the coordinate r of the skateboard is constant, $r = R$, and the position of the skateboard is completely specified by the angle ϕ. With r constant, the second law (1.48) takes the relatively simple form

$$F_r = -mR\dot{\phi}^2 \tag{1.49}$$

and

$$F_\phi = mR\ddot{\phi}. \tag{1.50}$$

The two forces on the skateboard are its weight $\mathbf{w} = m\mathbf{g}$ and the normal force $\mathbf{N}$ of the wall, as shown in Figure 1.14. The components of the net force $\mathbf{F} = \mathbf{w} + \mathbf{N}$ are easily seen to be

$$F_r = mg\cos\phi - N \quad \text{and} \quad F_\phi = -mg\sin\phi.$$

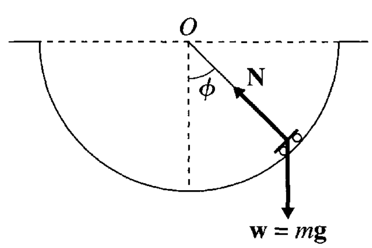

Figure 1.14 A skateboard in a semicircular trough of radius R. The board's position is specified by the angle ϕ measured up from the bottom. The two forces on the skateboard are its weight $\mathbf{w} = m\mathbf{g}$ and the normal force $\mathbf{N}$.

Substituting for F_r into (1.49) we get an equation involving N, ϕ, and $\dot{\phi}$. Fortunately, we are not really interested in N, and — even more fortunately — when we substitute for F_ϕ into (1.50), we get an equation that does not involve N at all:

$$-mg\sin\phi = mR\ddot{\phi}$$

or, canceling the m's and rearranging,

$$\ddot{\phi} = -\frac{g}{R}\sin\phi. \tag{1.51}$$

Equation (1.51) is the differential equation for $\phi(t)$ that determines the motion of the skateboard. Qualitatively, we can easily see the kind of motion that it implies. First, if $\phi = 0$, (1.51) says that $\ddot{\phi} = 0$. Therefore, if we place the board at rest ($\dot{\phi} = 0$) at the point $\phi = 0$, the board will never move (unless someone pushes it); that is, $\phi = 0$ is an equilibrium position, as you would certainly have guessed. Next, suppose that at some time, ϕ is not zero and, to be definite, suppose that $\phi > 0$; that is, the skateboard is on the right-hand side of the half-pipe. In this case, (1.51) implies that $\ddot{\phi} < 0$, so the acceleration is directed to the left. If the board is moving to the right it must slow down and eventually start moving to the left.[13] Once it is moving toward the left, it speeds up and returns to the bottom, where it moves over to the left. As soon as the board is on the left, the argument reverses ($\phi < 0$, so $\ddot{\phi} > 0$) and the board must eventually return to the bottom and move over to the right again. In other words, the differential equation (1.51) implies that the skateboard oscillates back and forth, from right to left and back to the right.

The equation of motion (1.51) cannot be solved in terms of elementary functions, such as polynomials, trigonometric functions, or logs and exponentials.[14] Thus, if we want more quantitative information about the motion, the simplest course is to use a computer to solve it numerically (see Problem 1.50). However, if the initial angle ϕ_o is *small*, we can use the small angle approximation

$$\sin\phi \approx \phi \tag{1.52}$$

and, within this approximation, (1.51) becomes

$$\ddot{\phi} = -\frac{g}{R}\phi \tag{1.53}$$

which *can* be solved using elementary functions. [By this stage, you have almost certainly recognized that our discussion of the skateboard problem closely parallels the analysis of the simple pendulum. In particular, the small-angle

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[13] I am taking for granted that it doesn't reach the top and jump out of the trough. Since it was released from rest inside the trough, this is correct. Much the easiest way to prove this claim is to invoke conservation of energy, which we shan't be discussing for a while. Perhaps, for now, you could agree to accept it as a matter of common sense.

[14] Actually the solution of (1.51) is a Jacobi elliptic function. However, I shall take the point of view that for most of us the Jacobi function is not "elementary."

approximation (1.52) is what let you solve the simple pendulum in your introductory physics course. This parallel is, of course, no accident. Mathematically the two problems are exactly equivalent.] If we define the parameter

$$\omega = \sqrt{\frac{g}{R}}, \tag{1.54}$$

then (1.53) becomes

$$\ddot{\phi} = -\omega^2 \phi. \tag{1.55}$$

This is the equation of motion for our skateboard in the small-angle approximation. I would like to discuss its solution in some detail to introduce several ideas that we'll be using again and again in what follows. (If you've studied differential equations before, just see the next three paragraphs as a quick review.)

We first observe that it is easy to find two solutions of the equation (1.55) by inspection (that is, by inspired guessing). The function $\phi(t) = A\sin(\omega t)$ is clearly a solution for any value of the constant A. [Differentiating $\sin(\omega t)$ brings out a factor of ω and changes the sin to a cos; differentiating it again brings out another ω and changes the cos back to $-$sin. Thus the proposed solution does satisfy $\ddot{\phi} = -\omega^2\phi$.] Similarly, the function $\phi(t) = B\cos(\omega t)$ is another solution for any constant B. Furthermore, as you can easily check, the sum of these two solutions is itself a solution. Thus we have now found a whole family of solutions:

$$\phi(t) = A\sin(\omega t) + B\cos(\omega t) \tag{1.56}$$

is a solution for any values of the two constants A and B.

I now want to argue that *every* solution of the equation of motion (1.55) has the form (1.56). In other words, (1.56) is the *general solution* — we have found *all* solutions, and we need seek no further. To get some idea of why this is, note that the differential equation (1.55) is a statement about the second derivative $\ddot{\phi}$ of the unknown ϕ. Now, if we had actually been told what $\ddot{\phi}$ is, then we know from elementary calculus that we could find ϕ by two integrations, and the result would contain two unknown constants — the two constants of integration — that would have to be determined by looking (for example) at the initial values of ϕ and $\dot{\phi}$. In other words, knowledge of $\ddot{\phi}$ would tell us that ϕ itself is one of a family of functions containing precisely two undetermined constants. Of course, the differential equation (1.55) does not actually tell us $\ddot{\phi}$ — it is an equation for $\ddot{\phi}$ in terms of ϕ. Nevetheless, it is plausible that such an equation would imply that ϕ is one of a family of functions that contain precisely two undetermined constants. If you have studied differential equations, you know that this is the case; if you have not, then I must ask you to accept it as a plausible fact: For any given second-order differential equation [in a large class of "reasonable" equations, including (1.55) and all of the equations we shall encounter in this book], the solutions all belong to a family of functions containing precisely two independent constants — like the constants A and B in (1.56). (More generally, the solutions of an nth-order equation contain precisely n independent constants.)

This theorem sheds a new light on our solution (1.56). We already knew that any function of the form (1.56) is a solution of the equation of motion. Our theorem now guarantees that *every* solution of the equation of motion is of this form. This same argument applies to all the second-order differential equations we shall encounter. If, by hook or by crook, we can find a solution like (1.56) involving two arbitrary constants, then we are guaranteed that we have found the general solution of our equation.

All that remains is to pin down the two constants A and B for our skateboard. To do so, we must look at the initial conditions. At $t = 0$, Equation (1.56) implies that $\phi = B$. Therefore B is just the initial value of ϕ, which we are calling ϕ_o, so $B = \phi_o$. At $t = 0$, Equation (1.56) implies that $\dot{\phi} = \omega A$. Since I released the board from rest, this means that $A = 0$, and our solution is

$$\phi(t) = \phi_o \cos(\omega t). \tag{1.57}$$

The first thing to note about this solution is that, as we anticipated on general grounds, $\phi(t)$ oscillates, moving from positive to negative and back to positive periodically and indefinitely. In particular, the board first returns to its initial position ϕ_o when $\omega t = 2\pi$. The time that this takes is called the period of the motion and is denoted τ. Thus our conclusion is that the period of the skateboard's oscillations is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}. \tag{1.58}$$

We were given that $R = 5\,\text{m}$, and $g = 9.8\,\text{m/s}^2$. Substituting these numbers, we conclude that the skateboard returns to its starting point in a time $\tau = 4.5$ seconds.

Principal Definitions and Equations of Chapter 1

Dot and Cross Products

$$\mathbf{r} \cdot \mathbf{s} = rs\cos\theta = r_x s_x + r_y s_y + r_z s_z \qquad \text{[Eqs. (1.6) \& (1.7)]}$$

$$\mathbf{r} \times \mathbf{s} = (r_y s_z - r_z s_y,\ r_z s_x - r_x s_z,\ r_x s_y - r_y s_x) = \det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix} \qquad \text{[Eq. (1.9)]}$$

Inertial Frames

An inertial frame is any reference frame in which Newton's first law holds, that is, a nonaccelerating, nonrotating frame.

Unit Vectors of a Coordinate System

If (ξ, η, ζ) are an orthogonal system of coordinates, then

$$\hat{\boldsymbol{\xi}} = \text{ unit vector in direction of increasing } \xi \text{ with } \eta \text{ and } \zeta \text{ fixed}$$

and so on, and any vector $\mathbf{s}$ can be expanded as $\mathbf{s} = s_\xi \hat{\boldsymbol{\xi}} + s_\eta \hat{\boldsymbol{\eta}} + s_\zeta \hat{\boldsymbol{\zeta}}$.

Newton's Second Law in Various Coordinate Systems

Vector Form	Cartesian (x, y, z)	2D Polar (r, ϕ)	Cylindrical Polar (ρ, ϕ, z)
$\mathbf{F} = m\ddot{\mathbf{r}}$	$\begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$	$\begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases}$	$\begin{cases} F_r = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{z} \end{cases}$
	Eq. (1.35)	Eq. (1.48)	Problem 1.47 or 1.48

Problems for Chapter 1

The problems for each chapter are arranged according to section number. A problem listed for a given section requires an understanding of that section and earlier sections, but not of later sections. Within each section problems are listed in approximate order of difficulty. A single star (⋆) indicates straightforward problems involving just one main concept. Two stars (⋆⋆) identify problems that are slightly more challenging and usually involve more than one concept. Three stars (⋆⋆⋆) indicate problems that are distinctly more challenging, either because they are intrinsically difficult or involve lengthy calculations. Needless to say, these distinctions are hard to draw and are only approximate.

Problems that need the use of a computer are flagged thus: [Computer]. These are mostly classified as ⋆⋆⋆ on the grounds that it usually takes a long time to set up the necessary code — especially if you're just learning the language.

SECTION 1.2 Space and Time

1.1 ⋆ Given the two vectors $\mathbf{b} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$ find $\mathbf{b} + \mathbf{c}$, $5\mathbf{b} + 2\mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

1.2 ⋆ Two vectors are given as $\mathbf{b} = (1, 2, 3)$ and $\mathbf{c} = (3, 2, 1)$. (Remember that these statements are just a compact way of giving you the components of the vectors.) Find $\mathbf{b} + \mathbf{c}$, $5\mathbf{b} - 2\mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

1.3 ⋆ By applying Pythagoras's theorem (the usual two-dimensional version) twice over, prove that the length r of a three-dimensional vector $\mathbf{r} = (x, y, z)$ satisfies $r^2 = x^2 + y^2 + z^2$.

1.4 ★ One of the many uses of the scalar product is to find the angle between two given vectors. Find the angle between the vectors $\mathbf{b} = (1, 2, 4)$ and $\mathbf{c} = (4, 2, 1)$ by evaluating their scalar product.

1.5 ★ Find the angle between a body diagonal of a cube and any one of its face diagonals. [*Hint:* Choose a cube with side 1 and with one corner at O and the opposite corner at the point $(1, 1, 1)$. Write down the vector that represents a body diagonal and another that represents a face diagonal, and then find the angle between them as in Problem 1.4.]

1.6 ★ By evaluating their dot product, find the values of the scalar s for which the two vectors $\mathbf{b} = \hat{\mathbf{x}} + s\hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} - s\hat{\mathbf{y}}$ are orthogonal. (Remember that two vectors are orthogonal if and only if their dot product is zero.) Explain your answers with a sketch.

1.7 ★ Prove that the two definitions of the scalar product $\mathbf{r} \cdot \mathbf{s}$ as $rs\cos\theta$ (1.6) and $\sum r_i s_i$ (1.7) are equal. One way to do this is to choose your x axis along the direction of $\mathbf{r}$. [Strictly speaking you should first make sure that the definition (1.7) is independent of the choice of axes. If you like to worry about such niceties, see Problem 1.16.]

1.8 ★ **(a)** Use the definition (1.7) to prove that the scalar product is distributive, that is, $\mathbf{r} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{v}$. **(b)** If $\mathbf{r}$ and $\mathbf{s}$ are vectors that depend on time, prove that the product rule for differentiating products applies to $\mathbf{r} \cdot \mathbf{s}$, that is, that

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{s}) = \mathbf{r} \cdot \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{s}.$$

1.9 ★ In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, and c, that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the angle between the sides a and b. Show that the law of cosines is an immediate consequence of the identity $(\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$.

1.10 ★ A particle moves in a circle (center O and radius R) with constant angular velocity ω counterclockwise. The circle lies in the xy plane and the particle is on the x axis at time $t = 0$. Show that the particle's position is given by

$$\mathbf{r}(t) = \hat{\mathbf{x}}R\cos(\omega t) + \hat{\mathbf{y}}R\sin(\omega t).$$

Find the particle's velocity and acceleration. What are the magnitude and direction of the acceleration? Relate your results to well-known properties of uniform circular motion.

1.11 ★ The position of a moving particle is given as a function of time t to be

$$\mathbf{r}(t) = \hat{\mathbf{x}}b\cos(\omega t) + \hat{\mathbf{y}}c\sin(\omega t),$$

where b, c, and ω are constants. Describe the particle's orbit.

1.12 ★ The position of a moving particle is given as a function of time t to be

$$\mathbf{r}(t) = \hat{\mathbf{x}}b\cos(\omega t) + \hat{\mathbf{y}}c\sin(\omega t) + \hat{\mathbf{z}}v_o t$$

where b, c, v_o and ω are constants. Describe the particle's orbit.

1.13 ★ Let $\mathbf{u}$ be an arbitrary fixed unit vector and show that any vector $\mathbf{b}$ satisfies

$$b^2 = (\mathbf{u} \cdot \mathbf{b})^2 + (\mathbf{u} \times \mathbf{b})^2.$$

Explain this result in words, with the help of a picture.

1.14 ⋆ Prove that for any two vectors **a** and **b**,

$$|\mathbf{a} + \mathbf{b}| \leq (a + b).$$

[*Hint:* Work out $|\mathbf{a} + \mathbf{b}|^2$ and compare it with $(a + b)^2$.] Explain why this is called the triangle inequality.

1.15 ⋆ Show that the definition (1.9) of the cross product is equivalent to the elementary definition that $\mathbf{r} \times \mathbf{s}$ is perpendicular to both **r** and **s**, with magnitude $rs \sin\theta$ and direction given by the right-hand rule. [*Hint:* It is a fact (though quite hard to prove) that the definition (1.9) is independent of your choice of axes. Therefore you can choose axes so that **r** points along the x axis and **s** lies in the xy plane.]

1.16 ⋆⋆ **(a)** Defining the scalar product $\mathbf{r} \cdot \mathbf{s}$ by Equation (1.7), $\mathbf{r} \cdot \mathbf{s} = \sum r_i s_i$, show that Pythagoras's theorem implies that the magnitude of any vector **r** is $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$. **(b)** It is clear that the length of a vector does not depend on our choice of coordinate axes. Thus the result of part (a) guarantees that the scalar product $\mathbf{r} \cdot \mathbf{r}$, as defined by (1.7), is the same for any choice of orthogonal axes. Use this to prove that $\mathbf{r} \cdot \mathbf{s}$, as defined by (1.7), is the same for any choice of orthogonal axes. [*Hint:* Consider the length of the vector $\mathbf{r} + \mathbf{s}$.]

1.17 ⋆⋆ **(a)** Prove that the vector product $\mathbf{r} \times \mathbf{s}$ as defined by (1.9) is distributive; that is, that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) + (\mathbf{r} \times \mathbf{v})$. **(b)** Prove the product rule

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}.$$

Be careful with the order of the factors.

1.18 ⋆⋆ The three vectors **a**, **b**, **c** are the three sides of the triangle ABC with angles α, β, γ as shown in Figure 1.15. **(a)** Prove that the area of the triangle is given by any one of these three expressions:

$$\text{area} = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| = \tfrac{1}{2}|\mathbf{b} \times \mathbf{c}| = \tfrac{1}{2}|\mathbf{c} \times \mathbf{a}|.$$

(b) Use the equality of these three expressions to prove the so-called law of sines, that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

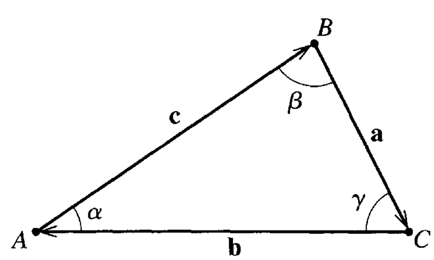

Figure 1.15 Triangle for Problem 1.18.

1.19 ★★ If $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}).$$

1.20 ★★ The three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ point from the origin O to the three corners of a triangle. Use the result of Problem 1.18 to show that the area of the triangle is given by

$$(\text{area of triangle}) = \tfrac{1}{2}|(\mathbf{B} \times \mathbf{C}) + (\mathbf{C} \times \mathbf{A}) + (\mathbf{A} \times \mathbf{B})|.$$

1.21 ★★ A parallelepiped (a six-faced solid with opposite faces parallel) has one corner at the origin O and the three edges that emanate from O defined by vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Show that the volume of the parallelepiped is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

1.22 ★★ The two vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the xy plane and make angles α and β with the x axis. **(a)** By evaluating $\mathbf{a} \cdot \mathbf{b}$ in two ways [namely using (1.6) and (1.7)] prove the well-known trig identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(b) By similarly evaluating $\mathbf{a} \times \mathbf{b}$ prove that

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

1.23 ★★ The unknown vector $\mathbf{v}$ satisfies $\mathbf{b} \cdot \mathbf{v} = \lambda$ and $\mathbf{b} \times \mathbf{v} = \mathbf{c}$, where λ, $\mathbf{b}$, and $\mathbf{c}$ are fixed and known. Find $\mathbf{v}$ in terms of λ, $\mathbf{b}$, and $\mathbf{c}$.

SECTION 1.4 Newton's First and Second Laws; Inertial Frames

1.24 ★ In case you haven't studied any differential equations before, I shall be introducing the necessary ideas as needed. Here is a simple excercise to get you started: Find the general solution of the first-order equation $df/dt = f$ for an unknown function $f(t)$. [There are several ways to do this. One is to rewrite the equation as $df/f = dt$ and then integrate both sides.] How many arbitrary constants does the general solution contain? [Your answer should illustrate the important general theorem that the solution to any nth-order differential equation (in a very large class of "reasonable" equations) contains n arbitrary constants.]

1.25 ★ Answer the same questions as in Problem 1.24, but for the differential equation $df/dt = -3f$.

1.26 ★★ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following: I am standing on a level floor at the origin of an inertial frame $\mathcal{S}$ and kick a frictionless puck due north across the floor. **(a)** Write down the x and y coordinates of the puck as functions of time as seen from my inertial frame. (Use x and y axes pointing east and north respectively.) Now consider two more observers, the first at rest in a frame $\mathcal{S}'$ that travels with constant velocity v due east relative to $\mathcal{S}$, the second at rest in a frame $\mathcal{S}''$ that travels with constant *acceleration* due east relative to $\mathcal{S}$. (All three frames coincide at the moment when I kick the puck, and $\mathcal{S}''$ is at rest relative to $\mathcal{S}$ at that same moment.) **(b)** Find the coordinates x', y' of the puck and describe the puck's path as seen from $\mathcal{S}'$. **(c)** Do the same for $\mathcal{S}''$. Which of the frames is inertial?

1.27 ★★ The hallmark of an inertial reference frame is that any object which is subject to zero net force will travel in a straight line at constant speed. To illustrate this, consider the following experiment: I am standing on the ground (which we shall take to be an inertial frame) beside a perfectly flat horizontal turntable, rotating with constant angular velocity ω. I lean over and shove a frictionless puck so that it slides across the turntable, straight through the center. The puck is subject to zero net force and, as seen from my inertial frame, travels in a straight line. Describe the puck's path as observed by someone sitting at rest on the turntable. This requires careful thought, but you should be able to get a qualitative picture. For a quantitative picture, it helps to use polar coordinates; see Problem 1.46.

SECTION 1.5 The Third Law and Conservation of Momentum

1.28 ⋆ Go over the steps from Equation (1.25) to (1.29) in the proof of conservation of momentum, but treat the case that $N = 3$ and write out all the summations explicitly to be sure you understand the various manipulations.

1.29 ⋆ Do the same tasks as in Problem 1.28 but for the case of four particles ($N = 4$).

1.30 ⋆ Conservation laws, such as conservation of momentum, often give a surprising amount of information about the possible outcome of an experiment. Here is perhaps the simplest example: Two objects of masses m_1 and m_2 are subject to no external forces. Object 1 is traveling with velocity $\mathbf{v}$ when it collides with the stationary object 2. The two objects stick together and move off with common velocity $\mathbf{v}'$. Use conservation of momentum to find $\mathbf{v}'$ in terms of $\mathbf{v}$, m_1, and m_2.

1.31 ⋆ In Section 1.5 we proved that Newton's third law implies the conservation of momentum. Prove the converse, that if the law of conservation of momentum applies to every possible group of particles, then the interparticle forces must obey the third law. [*Hint:* However many particles your system contains, you can focus your attention on just two of them. (Call them 1 and 2.) The law of conservation of momentum says that if there are no external forces on this pair of particles, then their total momentum must be constant. Use this to prove that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.]

1.32 ⋆⋆ If you have some experience in electromagnetism, you could do the following problem concerning the curious situation illustrated in Figure 1.8. The electric and magnetic fields at a point $\mathbf{r}_1$ due to a charge q_2 at $\mathbf{r}_2$ moving with constant velocity $\mathbf{v}_2$ (with $v_2 \ll c$) are[15]

$$\mathbf{E}(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_o}\frac{q_2}{s^2}\hat{\mathbf{s}} \quad \text{and} \quad \mathbf{B}(\mathbf{r}_1) = \frac{\mu_o}{4\pi}\frac{q_2}{s^2}\mathbf{v}_2 \times \hat{\mathbf{s}}$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector pointing from $\mathbf{r}_2$ to $\mathbf{r}_1$. (The first of these you should recognize as Coulomb's law.) If $\mathbf{F}_{12}^{\text{el}}$ and $\mathbf{F}_{12}^{\text{mag}}$ denote the electric and magnetic forces on a charge q_1 at $\mathbf{r}_1$ with velocity $\mathbf{v}_1$, show that $F_{12}^{\text{mag}} \leq (v_1 v_2/c^2)F_{12}^{\text{el}}$. This shows that in the non-relativistic domain it is legitimate to ignore the magnetic force between two moving charges.

1.33 ⋆⋆⋆ If you have some experience in electromagnetism and with vector calculus, prove that the magnetic forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, between two steady current loops obey Newton's third law. [*Hints:* Let the two currents be I_1 and I_2 and let typical points on the two loops be $\mathbf{r}_1$ and $\mathbf{r}_2$. If $d\mathbf{r}_1$ and $d\mathbf{r}_2$ are short segments of the loops, then according to the Biot–Savart law, the force on $d\mathbf{r}_1$ due to $d\mathbf{r}_2$ is

$$\frac{\mu_o}{4\pi}\frac{I_1 I_2}{s^2}\, d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$. The force $\mathbf{F}_{12}$ is found by integrating this around both loops. You will need to use the "$BAC - CAB$" rule to simplify the triple product.]

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[15] See, for example, David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Prentice Hall, (1999), p. 440.

1.34 ★★★ Prove that in the absence of external forces, the total *angular* momentum (defined as $\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha$) of an N-particle system is conserved. [*Hints*: You need to mimic the argument from (1.25) to (1.29). In this case you need more than Newton's third law: In addition you need to assume that the interparticle forces are *central*; that is, $\mathbf{F}_{\alpha\beta}$ acts along the line joining particles α and β. A full discussion of angular momentum is given in Chapter 3.]

SECTION 1.6 Newton's Second Law in Cartesian Coordinates

1.35 ★ A golf ball is hit from ground level with speed v_o in a direction that is due east and at an angle θ above the horizontal. Neglecting air resistance, use Newton's second law (1.35) to find the position as a function of time, using coordinates with x measured east, y north, and z vertically up. Find the time for the golf ball to return to the ground and how far it travels in that time.

1.36 ★ A plane, which is flying horizontally at a constant speed v_o and at a height h above the sea, must drop a bundle of supplies to a castaway on a small raft. **(a)** Write down Newton's second law for the bundle as it falls from the plane, assuming you can neglect air resistance. Solve your equations to give the bundle's position in flight as a function of time t. **(b)** How far before the raft (measured horizontally) must the pilot drop the bundle if it is to hit the raft? What is this distance if $v_o = 50$ m/s, $h = 100$ m, and $g \approx 10$ m/s^2? **(c)** Within what interval of time ($\pm\Delta t$) must the pilot drop the bundle if it is to land within ± 10 m of the raft?

1.37 ★ A student kicks a frictionless puck with initial speed v_o, so that it slides straight up a plane that is inclined at an angle θ above the horizontal. **(a)** Write down Newton's second law for the puck and solve to give its position as a function of time. **(b)** How long will the puck take to return to its starting point?

1.38 ★ You lay a rectangular board on the horizontal floor and then tilt the board about one edge until it slopes at angle θ with the horizontal. Choose your origin at one of the two corners that touch the floor, the x axis pointing along the bottom edge of the board, the y axis pointing up the slope, and the z axis normal to the board. You now kick a frictionless puck that is resting at O so that it slides across the board with initial velocity $(v_{ox}, v_{oy}, 0)$. Write down Newton's second law using the given coordinates and then find how long the puck takes to return to the floor level and how far it is from O when it does so.

1.39 ★★ A ball is thrown with initial speed v_o up an inclined plane. The plane is inclined at an angle ϕ above the horizontal, and the ball's initial velocity is at an angle θ above the plane. Choose axes with x measured up the slope, y normal to the slope, and z across it. Write down Newton's second law using these axes and find the ball's position as a function of time. Show that the ball lands a distance $R = 2v_o^2 \sin\theta\cos(\theta+\phi)/(g\cos^2\phi)$ from its launch point. Show that for given v_o and ϕ, the maximum possible range up the inclined plane is $R_{max} = v_o^2/[g(1+\sin\phi)]$.

1.40 ★★★ A cannon shoots a ball at an angle θ above the horizontal ground. **(a)** Neglecting air resistance, use Newton's second law to find the ball's position as a function of time. (Use axes with x measured horizontally and y vertically.) **(b)** Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ if $r(t)$ is to increase throughout the ball's flight? [*Hint:* Using your solution to part (a) you can write down r^2 as $x^2 + y^2$, and then find the condition that r^2 is always increasing.]

SECTION 1.7 Two-Dimensional Polar Coordinates

1.41 ⋆ An astronaut in gravity-free space is twirling a mass m on the end of a string of length R in a circle, with constant angular velocity ω. Write down Newton's second law (1.48) in polar coordinates and find the tension in the string.

1.42 ⋆ Prove that the transformations from rectangular to polar coordinates and vice versa are given by the four equations (1.37). Explain why the equation for ϕ is not quite complete and give a complete version.

1.43 ⋆ **(a)** Prove that the unit vector $\hat{\mathbf{r}}$ of two-dimensional polar coordinates is equal to

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi \tag{1.59}$$

and find a corresponding expression for $\hat{\boldsymbol{\phi}}$. **(b)** Assuming that ϕ depends on the time t, differentiate your answers in part (a) to give an alternative proof of the results (1.42) and (1.46) for the time derivatives $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\phi}}}$.

1.44 ⋆ Verify by direct substitution that the function $\phi(t) = A\sin(\omega t) + B\cos(\omega t)$ of (1.56) is a solution of the second-order differential equation (1.55), $\ddot{\phi} = -\omega^2\phi$. (Since this solution involves two arbitrary constants — the coefficients of the sine and cosine functions — it is in fact the general solution.)

1.45 ⋆⋆ Prove that if $\mathbf{v}(t)$ is any vector that depends on time (for example the velocity of a moving particle) but which has *constant magnitude*, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. Prove the converse that if $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$, then $|\mathbf{v}(t)|$ is constant. [*Hint:* Consider the derivative of $\mathbf{v}^2$.] This is a very handy result. It explains why, in two-dimensional polars, $d\hat{\mathbf{r}}/dt$ has to be in the direction of $\hat{\boldsymbol{\phi}}$ and vice versa. It also shows that the speed of a charged particle in a magnetic field is constant, since the acceleration is perpendicular to the velocity.

1.46 ⋆⋆ Consider the experiment of Problem 1.27, in which a frictionless puck is slid straight across a rotating turntable through the center O. **(a)** Write down the polar coordinates r, ϕ of the puck as functions of time, as measured in the inertial frame $\mathcal{S}$ of an observer on the ground. (Assume that the puck was launched along the axis $\phi = 0$ at $t = 0$.) **(b)** Now write down the polar coordinates r', ϕ' of the puck as measured by an observer (frame $\mathcal{S}'$) at rest on the turntable. (Choose these coordinates so that ϕ and ϕ' coincide at $t = 0$.) Describe and sketch the path seen by this second observer. Is the frame $\mathcal{S}'$ inertial?

1.47 ⋆⋆ Let the position of a point P in three dimensions be given by the vector $\mathbf{r} = (x, y, z)$ in rectangular (or Cartesian) coordinates. The same position can be specified by **cylindrical polar coordinates**, ρ, ϕ, z, which are defined as follows: Let P' denote the projection of P onto the xy plane; that is, P' has Cartesian coordinates $(x, y, 0)$. Then ρ and ϕ are defined as the two-dimensional polar coordinates of P' in the xy plane, while z is the third Cartesian coordinate, unchanged. **(a)** Make a sketch to illustrate the three cylindrical coordinates. Give expressions for ρ, ϕ, z in terms of the Cartesian coordinates x, y, z. Explain in words what ρ is ("ρ is the distance of P from ______"). There are many variants in notation. For instance, some people use r instead of ρ. Explain why this use of r is unfortunate. **(b)** Describe the three unit vectors $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and write the expansion of the position vector $\mathbf{r}$ in terms of these unit vectors. **(c)** Differentiate your last answer twice to find the cylindrical components of the acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle. To do this, you will need to know the time derivatives of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. You could get these from the corresponding two-dimensional results (1.42) and (1.46), or you could derive them directly as in Problem 1.48.

1.48 ⋆⋆ Find expressions for the unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ of cylindrical polar coordinates (Problem 1.47) in terms of the Cartesian $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$. Differentiate these expressions with respect to time to find $d\hat{\boldsymbol{\rho}}/dt$, $d\hat{\boldsymbol{\phi}}/dt$, and $d\hat{\mathbf{z}}/dt$.

1.49 ⋆⋆ Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

1.50 ⋆⋆⋆ [Computer] The differential equation (1.51) for the skateboard of Example 1.2 cannot be solved in terms of elementary functions, but is easily solved numerically. **(a)** If you have access to software, such as Mathematica, Maple, or Matlab, that can solve differential equations numerically, solve the differential equation for the case that the board is released from $\phi_o = 20$ degrees, using the values $R = 5$ m and $g = 9.8$ m/s^2. Make a plot of ϕ against time for two or three periods. **(b)** On the same picture, plot the approximate solution (1.57) with the same $\phi_o = 20°$. Comment on your two graphs. Note: If you haven't used the numerical solver before, you will need to learn the necessary syntax. For example, in Mathematica you will need to learn the syntax for "NDSolve" and how to plot the solution that it provides. This takes a bit of time, but is something that is very well worth learning.

1.51 ⋆⋆⋆ [Computer] Repeat all of Problem 1.50 but using the initial value $\phi_o = \pi/2$.

CHAPTER 2

Projectiles and Charged Particles

In this chapter, I present two topics: the motion of projectiles subject to the forces of gravity and air resistance, and the motion of charged particles in uniform magnetic fields. Both problems lend themselves to solution using Newton's laws in Cartesian coordinates, and both allow us to review and introduce some important mathematics. Above all, both are problems of great practical interest.

2.1 Air Resistance

Most introductory physics courses spend some time studying the motion of projectiles, but they almost always ignore air resistance. In many problems this is an excellent approximation; in others, air resistance is obviously important, and we need to know how to account for it. More generally, whether or not air resistance is significant, we need some way to estimate how important it really is.

Let us begin by surveying some of the basic properties of the resistive force, or **drag**, $\mathbf{f}$ of the air, or other medium, through which an object is moving. (I shall generally speak of "air resistance" since air is the medium through which most projectiles move, but the same considerations apply to other gases and often to liquids as well.) The most obvious fact about air resistance, well known to anyone who rides a bicycle, is that it depends on the speed, v, of the object concerned. In addition, for many objects, the direction of the force due to motion through the air is opposite to the velocity $\mathbf{v}$. For certain objects, such as a nonrotating sphere, this is exactly true, and for many it is a good approximation. You should, however, be aware that there are situations where it is certainly not true: The force of the air on an airplane wing has a large sideways component, called the **lift**, without which no airplanes could fly. Nevertheless, I shall assume that $\mathbf{f}$ and $\mathbf{v}$ point in opposite directions; that is, I shall consider only objects for which the sideways force is zero, or at least small enough

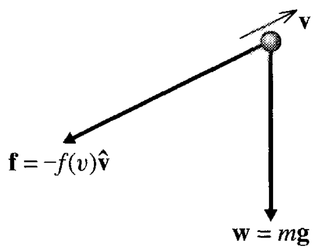

Figure 2.1 A projectile is subject to two forces, the force of gravity, $\mathbf{w} = m\mathbf{g}$, and the drag force of air resistance, $\mathbf{f} = -f(v)\hat{\mathbf{v}}$.

to be neglected. The situation is illustrated in Figure 2.1 and is summed up in the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}, \tag{2.1}$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ denotes the unit vector in the direction of $\mathbf{v}$, and $f(v)$ is the magnitude of $\mathbf{f}$.

The function $f(v)$ that gives the magnitude of the air resistance varies with v in a complicated way, especially as the object's speed approaches the speed of sound. However, at lower speeds it is often a good approximation to write[1]

$$f(v) = bv + cv^2 = f_{\text{lin}} + f_{\text{quad}} \tag{2.2}$$

where f_{lin} and f_{quad} stand for the linear and quadratic terms respectively,

$$f_{\text{lin}} = bv \quad \text{and} \quad f_{\text{quad}} = cv^2. \tag{2.3}$$

The physical origins of these two terms are quite different: The linear term, f_{lin}, arises from the viscous drag of the medium and is generally proportional to the viscosity of the medium and the linear size of the projectile (Problem 2.2). The quadratic term, f_{quad}, arises from the projectile's having to accelerate the mass of air with which it is continually colliding; f_{quad} is proportional to the density of the medium and the cross-sectional area of the projectile (Problem 2.4). In particular, for a spherical projectile (a cannonball, a baseball, or a drop of rain), the coefficients b and c in (2.2) have the form

$$b = \beta D \quad \text{and} \quad c = \gamma D^2 \tag{2.4}$$

where D denotes the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. For a spherical projectile in air at STP, they have the approximate values

$$\beta = 1.6 \times 10^{-4}\ \text{N·s/m}^2 \tag{2.5}$$

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[1] Mathematically, Equation (2.2) is, in a sense, obvious. Any reasonable function is expected to have a Taylor series expansion, $f = a + bv + cv^2 + \cdots$. For low enough v, the first three terms should give a good approximation, and, since $f = 0$ when $v = 0$ the constant term, a, has to be zero.

and

$$\gamma = 0.25 \text{ N}\cdot\text{s}^2/\text{m}^4. \tag{2.6}$$

(For calculation of these two constants, see Problems 2.2 and 2.4.) You need to remember that these values are valid only for a sphere moving through air at STP. Nevertheless, they give at least a rough idea of the importance of the drag force even for nonspherical bodies moving through different gases at any normal temperatures and pressures.

It often happens that we can neglect one of the terms in (2.2) compared to the other, and this simplifies the task of solving Newton's second law. To decide whether this does happen in a given problem, and which term to neglect, we need to compare the sizes of the two terms:

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta}v = \left(1.6 \times 10^3 \,\frac{\text{s}}{\text{m}^2}\right) Dv \tag{2.7}$$

if we use the values (2.5) and (2.6) for a sphere in air. In a given problem, we have only to substitute the values of D and v into this equation to find out if one of the terms can be neglected, as the following example illustrates.

EXAMPLE 2.1 A Baseball and Some Drops of Liquid

Assess the relative importance of the linear and quadratic drags on a baseball of diameter $D = 7$ cm, traveling at a modest $v = 5$ m/s. Do the same for a drop of rain ($D = 1$ mm and $v = 0.6$ m/s) and for a tiny droplet of oil used in the Millikan oildrop experiment ($D = 1.5\,\mu$m and $v = 5 \times 10^{-5}$ m/s).

When we substitute the numbers for the baseball into (2.7) (remembering to convert the diameter to meters), we get

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 600 \qquad \text{[baseball]}. \tag{2.8}$$

For this baseball, the linear term is clearly negligible and we need consider only the quadratic drag. If the ball is traveling faster, the ratio $f_{\text{quad}}/f_{\text{lin}}$ is even greater. At slower speeds the ratio is less dramatic, but even at 1 m/s the ratio is 100. In fact if v is small enough that the linear term is comparable to the quadratic, both terms are so small as to be negligible. Thus, for baseballs and similar objects, it is almost always safe to neglect f_{lin} and take the drag force to be

$$\mathbf{f} = -cv^2\hat{\mathbf{v}}. \tag{2.9}$$

For the raindrop, the numbers give

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 1 \qquad \text{[raindrop]}. \tag{2.10}$$

Thus for this raindrop the two terms are comparable and neither can be neglected — which makes solving for the motion more difficult. If the drop were a lot larger or were traveling much faster, then the linear term would be negligible; and if the drop were much smaller or were traveling much slower, then the quadratic term would be negligible. But in general, with raindrops and similar objects, we are going to have to take both f_{lin} and f_{quad} into account.

For the oildrop in the Millikan experiment the numbers give

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 10^{-7} \qquad \text{[Millikan oildrop].} \tag{2.11}$$

In this case, the quadratic term is totally negligible, and we can take

$$\mathbf{f} = -bv\hat{\mathbf{v}} = -b\mathbf{v}, \tag{2.12}$$

where the second, very compact form follows because, of course, $v\hat{\mathbf{v}} = \mathbf{v}$.

The moral of this example is clear: First, there are objects for which the drag force is dominantly linear, and the quadratic force can be neglected — notably, very small liquid drops in air, but also slightly larger objects in a very viscous fluid, such as a ball bearing moving through molasses. On the other hand, for most projectiles, such as golf balls, cannonballs, and even a human in free fall, the dominant drag force is quadratic, and we can neglect the linear term. This situation is a little unlucky because the linear problem is much easier to solve than the quadratic. In the following two sections, I shall discuss the linear case, precisely because it is the easier one. Nevertheless, it *does* have practical applications, and the mathematics used to solve it is widely used in many fields. In Section 2.4, I shall take up the harder but more usual case of quadratic drag.

To conclude this introductory section, I should mention the Reynolds number, an important parameter that features prominently in more advanced treatments of motion in fluids. As already mentioned, the linear drag f_{lin} can be related to the viscosity of the fluid through which our projectile is moving, and the quadratic term f_{quad} is similarly related to the inertia (and hence density) of the fluid. Thus one can relate the ratio $f_{\text{quad}}/f_{\text{lin}}$ to the fundamental parameters η, the viscosity, and ϱ, the density, of the fluid (see Problem 2.3). The result is that the ratio $f_{\text{quad}}/f_{\text{lin}}$ is of roughly the same order of magnitude as the dimensionless number $R = Dv\varrho/\eta$, called the **Reynolds number**. Thus a compact and general way to summarize the foregoing discussion is to say that the quadratic drag f_{quad} is dominant when the Reynolds number R is large, whereas the linear drag dominates when R is small.

2.2 Linear Air Resistance

Let us consider first a projectile for which the quadratic drag force is negligible, so that the force of air resistance is given by (2.12). We shall see directly that, because the drag force is linear in $\mathbf{v}$, the equations of motion are very simple to solve. The two forces on the projectile are the weight $\mathbf{w} = m\mathbf{g}$ and the drag force $\mathbf{f} = -b\mathbf{v}$, as shown in Figure 2.2. Thus the second law, $m\ddot{\mathbf{r}} = \mathbf{F}$, reads

$$m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v}. \tag{2.13}$$

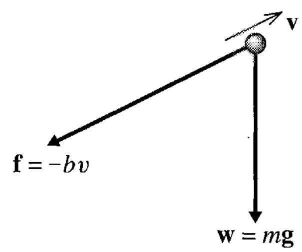

Figure 2.2 The two forces on a projectile for which the force of air resistance is linear in the velocity, $\mathbf{f} = -b\mathbf{v}$.

An interesting feature of this form is that, because neither of the forces depends on $\mathbf{r}$, the equation of motion does not involve $\mathbf{r}$ itself (only the first and second derivatives of $\mathbf{r}$). In fact, we can rewrite $\ddot{\mathbf{r}}$ as $\dot{\mathbf{v}}$, and (2.13) becomes

$$m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v}, \tag{2.14}$$

a first-order differential equation for $\mathbf{v}$. This simplification comes about because the forces depend only on $\mathbf{v}$ and not $\mathbf{r}$. It means we have to solve only a first-order differential equation for $\mathbf{v}$ and then integrate $\mathbf{v}$ to find $\mathbf{r}$.

Perhaps the most important simplifying feature of linear drag is that the equation of motion separates into components especially easily. For instance, with x measured to the right and y vertically downward, (2.14) resolves into

$$m\dot{v}_x = -bv_x \tag{2.15}$$

and

$$m\dot{v}_y = mg - bv_y. \tag{2.16}$$

That is, we have two separate equations, one for v_x and one for v_y; the equation for v_x does not involve v_y and vice versa. It is important to recognize that this happened only because the drag force was linear in $\mathbf{v}$. For instance, if the drag force were quadratic,

$$\mathbf{f} = -cv^2\hat{\mathbf{v}} = -cv\mathbf{v} = -c\sqrt{v_x^2 + v_y^2}\,\mathbf{v}, \tag{2.17}$$

then in (2.14) we would have to replace the term $-b\mathbf{v}$ with (2.17). In place of the two equations (2.15) and (2.16), we would have

$$\left.\begin{aligned} m\dot{v}_x &= -c\sqrt{v_x^2 + v_y^2}\,v_x \\ m\dot{v}_y &= mg - c\sqrt{v_x^2 + v_y^2}\,v_y. \end{aligned}\right\} \tag{2.18}$$

Here, each equation involves *both* of the variables v_x and v_y. These two *coupled* differential equations are much harder to solve than the uncoupled equations of the linear case.

Because they are uncoupled, we can solve each equation for linear drag separately and then put the two solutions together. Further, each equation defines a problem that is interesting in its own right. Equation (2.15) is the equation of motion for an object

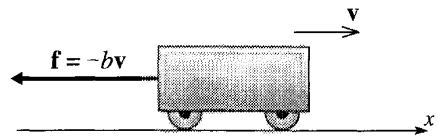

Figure 2.3 A cart moves on a horizontal frictionless track in a medium that produces a linear drag force.

(a cart with frictionless wheels, for instance) coasting horizontally in a medium that causes linear drag. Equation (2.16) describes an object (a tiny oil droplet for instance) that is falling vertically with linear air resistance. I shall solve these two separate problems in turn.

Horizontal Motion with Linear Drag

Consider an object such as the cart in Figure 2.3 coasting horizontally in a linearly resistive medium. I shall assume that at $t = 0$ the cart is at $x = 0$ with velocity $v_x = v_{xo}$. The only force on the cart is the drag $\mathbf{f} = -b\mathbf{v}$, thus the cart inevitably slows down. The rate of slowing is determined by (2.15), which has the general form

$$\dot{v}_x = -kv_x, \tag{2.19}$$

where k is my temporary abbreviation for $k = b/m$. This is a first-order differential equation for v_x, whose general solution must contain exactly one arbitrary constant. The equation states that the derivative of v_x is equal to $-k$ times v_x itself, and the only function with this property is the exponential function

$$v_x(t) = Ae^{-kt} \tag{2.20}$$

which satisfies (2.19) for any value of the constant A (Problems 1.24 and 1.25). Since this solution contains one arbitrary constant, it is the *general* solution of our first-order equation; that is, *any* solution must have this form. In our case, we know that $v_x(0) = v_{xo}$, so that $A = v_{xo}$, and we conclude that

$$v_x(t) = v_{xo}e^{-kt} = v_{xo}e^{-t/\tau}, \tag{2.21}$$

where I have introduced the convenient parameter

$$\tau = 1/k = m/b \qquad \text{[for linear drag]}. \tag{2.22}$$

We see that our cart slows down exponentially, as shown in Figure 2.4(a). The parameter τ has the dimensions of time (as you should check), and you can see from (2.21) that when $t = \tau$, the velocity is $1/e$ of its initial value; that is, τ is the "$1/e$" time for the exponentially decreasing velocity. As $t \to \infty$, the velocity approaches zero.

To find the position as a function of time, we have only to integrate the velocity (2.21). Integrations of this kind can be done using the definite or indefinite integral. The definite integral has the advantage that it automatically takes care of the constant

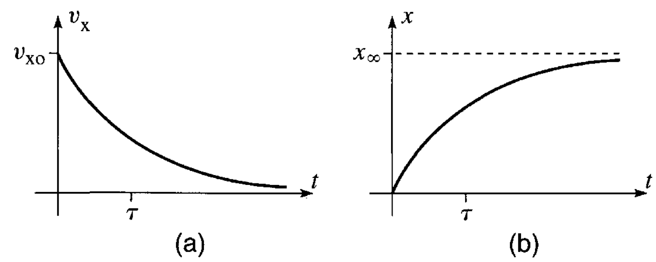

Figure 2.4 **(a)** The velocity v_x as a function of time, t, for a cart moving horizontally with a linear resistive force. As $t \to \infty$, v_x approaches zero exponentially. **(b)** The position x as a function of t for the same cart. As $t \to \infty$, $x \to x_\infty = v_{xo}\tau$.

of integration: Since $v_x = dx/dt$,

$$\int_0^t v_x(t')\, dt' = x(t) - x(0).$$

(Notice that I have named the "dummy" variable of integration t' to avoid confusion with the upper limit t.) Therefore

$$\begin{aligned} x(t) &= x(0) + \int_0^t v_{xo} e^{-t'/\tau}\, dt' \\ &= 0 + \left[-v_{xo}\tau e^{-t'/\tau}\right]_0^t \\ &= x_\infty\left(1 - e^{-t/\tau}\right). \end{aligned} \tag{2.23}$$

In the second line, I have used our assumption that $x = 0$ when $t = 0$. And in the last, I have introduced the parameter

$$x_\infty = v_{xo}\tau, \tag{2.24}$$

which is the limit of $x(t)$ as $t \to \infty$. We conclude that, as the cart slows down, its position approaches x_∞ asymptotically, as shown in Figure 2.4(b).

Vertical Motion with Linear Drag

Let us next consider a projectile that is subject to linear air resistance and is thrown vertically downward. The two forces on the projectile are gravity and air resistance, as shown in Figure 2.5. If we measure y vertically down, the only interesting component of the equation of motion is the y component, which reads

$$m\dot{v}_y = mg - bv_y. \tag{2.25}$$

With the velocity downward ($v_y > 0$), the retarding force is upward, while the force of gravity is downward. If v_y is small, the force of gravity is more important than the drag force, and the falling object accelerates in its downward motion. This will

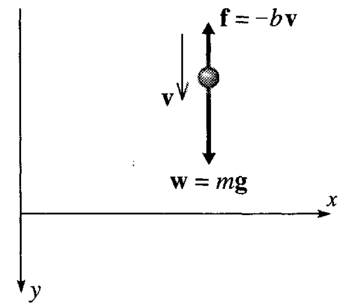

Figure 2.5 The forces on a projectile that is thrown vertically down, subject to linear air resistance.

continue until the drag force balances the weight. The speed at which this balance occurs is easily found by setting (2.25) equal to zero, to give $v_y = mg/b$ or

$$v_y = v_{\text{ter}}$$

where I have defined the **terminal speed**

$$v_{\text{ter}} = \frac{mg}{b} \qquad \text{[for linear drag].} \tag{2.26}$$

The terminal speed is the speed at which our projectile will eventually fall, if given the time to do so. Since it depends on m and b, it is different for different bodies. For example, if two objects have the same shape and size (b the same for both), the heavier object (m larger) will have the higher terminal speed, just as you would expect. Since v_{ter} is inversely proportional to the coefficient b of air resistance, we can view v_{ter} as an inverse measure of the importance of air resistance — the larger the air resistance, the smaller v_{ter}, again just as you would expect.

EXAMPLE 2.2 Terminal Speed of Small Liquid Drops

Find the terminal speed of a tiny oildrop in the Millikan oildrop experiment (diameter $D = 1.5\,\mu\text{m}$ and density $\varrho = 840\,\text{kg/m}^3$). Do the same for a small drop of mist with diameter $D = 0.2$ mm.

From Example 2.1 we know that the linear drag is dominant for these objects, so the terminal speed is given by (2.26). According to (2.4), $b = \beta D$ where $\beta = 1.6 \times 10^{-4}$ (in SI units). The mass of the drop is $m = \varrho\,\pi D^3/6$. Thus (2.26) becomes

$$v_{\text{ter}} = \frac{\varrho\,\pi D^2 g}{6\beta} \qquad \text{[for linear drag].} \tag{2.27}$$

This interesting result shows that, for a given density, the terminal speed is proportional to D^2. This implies that, once air resistance has become important, a large sphere will fall faster than a small sphere of the same density.[2]

Putting in the numbers, we find for the oildrop

$$v_{\text{ter}} = \frac{(840) \times \pi \times (1.5 \times 10^{-6})^2 \times (9.8)}{6 \times (1.6 \times 10^{-4})} = 6.1 \times 10^{-5}\,\text{m/s} \qquad \text{[oildrop]}.$$

In the Millikan oildrop experiment, the oildrops fall exceedingly slowly, so their speed can be measured by simply watching them through a microscope.

Putting in the numbers for the drop of mist, we find similarly that

$$v_{\text{ter}} = 1.3\,\text{m/s} \qquad \text{[drop of mist]}. \tag{2.28}$$

This speed is representative for a fine drizzle. For a larger raindrop, the terminal speed would be appreciably larger, but with a larger (and hence also faster) drop, the quadratic drag would need to be included in the calculation to get a reliable value for v_{ter}.

So far, we have discussed the terminal speed of a projectile (moving vertically), but we must now discuss how the projectile approaches that speed. This is determined by the equation of motion (2.25) which we can rewrite as

$$m\dot{v}_y = -b(v_y - v_{\text{ter}}). \tag{2.29}$$

(Remember that $v_{\text{ter}} = mg/b$.) This differential equation can be solved in several ways. (For one alternative see Problem 2.9.) Perhaps the simplest is to note that it is almost the same as Equation (2.15) for the horizontal motion, except that on the right we now have $(v_y - v_{\text{ter}})$ instead of v_x. The solution for the horizontal case was the exponential function (2.20). The trick to solving our new vertical equation (2.29) is to introduce the new variable $u = (v_y - v_{\text{ter}})$, which satisfies $m\dot{u} = -bu$ (because v_{ter} is constant). Since this is *exactly* the same as Equation (2.15) for the horizontal motion, the solution for u is the same exponential, $u = Ae^{-t/\tau}$. [Remember that the constant k in (2.20) became $k = 1/\tau$.] Therefore,

$$v_y - v_{\text{ter}} = Ae^{-t/\tau}.$$

When $t = 0$, $v_y = v_{yo}$, so $A = v_{yo} - v_{\text{ter}}$ and our final solution for v_y as a function of t is

$$v_y(t) = v_{\text{ter}} + (v_{yo} - v_{\text{ter}})e^{-t/\tau} \tag{2.30}$$

$$= v_{yo}e^{-t/\tau} + v_{\text{ter}}\left(1 - e^{-t/\tau}\right). \tag{2.31}$$

[2] We are here assuming that the drag force is linear, but the same qualitative conclusion follows for a quadratic drag force. (Problem 2.24.)

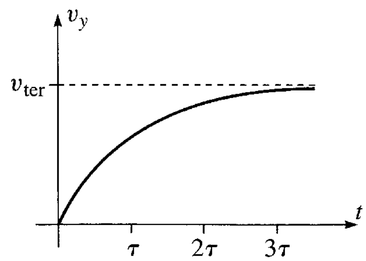

Figure 2.6 When an object is dropped in a medium with linear resistance, v_y approaches its terminal value v_{ter} as shown.

This second expression gives $v_y(t)$ as the sum of two terms: The first is equal to v_{yo} when $t = 0$, but fades away to zero as t increases; the second is equal to zero when $t = 0$, but approaches v_{ter} as $t \to \infty$. In particular, as $t \to \infty$,

$$v_y(t) \to v_{ter} \tag{2.32}$$

just as we anticipated.

Let us examine the result (2.31) in a little more detail for the case that $v_{yo} = 0$; that is, the projectile is dropped from rest. In this case (2.31) reads

$$v_y(t) = v_{ter}\left(1 - e^{-t/\tau}\right). \tag{2.33}$$

This result is plotted in Figure 2.6, where we see that v_y starts out from 0 and approaches the terminal speed, $v_y \to v_{ter}$, asymptotically as $t \to \infty$. The significance of the time τ for a falling body is easily read off from (2.33). When $t = \tau$, we see that

$$v_y = v_{ter}(1 - e^{-1}) = 0.63 v_{ter}.$$

That is, in a time τ, the object reaches 63% of the terminal speed. Similar calculations give the following results:

time t	percent of v_{ter}
0	0
τ	63%
2τ	86%
3τ	95%

Of course, the object's speed never actually reaches v_{ter}, but τ is a good measure of how fast the speed approaches v_{ter}. In particular, when $t = 3\tau$ the speed is 95% of v_{ter}, and for many purposes we can say that after a time 3τ the speed is essentially equal to v_{ter}.

EXAMPLE 2.3 Characteristic Time for Two Liquid Drops

Find the characteristic times, τ, for the oildrop and drop of mist in Example 2.2.

The characteristic time τ was defined in (2.22) as $\tau = m/b$, and $v_{\rm ter}$ was defined in (2.26) as $v_{\rm ter} = mg/b$. Thus we have the useful relation

$$v_{\rm ter} = g\tau. \tag{2.34}$$

Notice that this relation lets us interpret $v_{\rm ter}$ as the speed a falling object *would* acquire in a time τ, *if it had a constant acceleration equal to* g. Also note that, like $v_{\rm ter}$, the time τ is an inverse indicator of the importance of air resistance: When the coefficient b of air resistance is small, both $v_{\rm ter}$ and τ are large; when b is large, both $v_{\rm ter}$ and τ are small.

For our present purposes, the importance of (2.34) is that, since we have already found the terminal velocities of the two drops, we can immediately find the values of τ. For the Millikan oildrop, we found that $v_{\rm ter} = 6.1 \times 10^{-5}$ m/s, therefore

$$\tau = \frac{v_{\rm ter}}{g} = \frac{6.1 \times 10^{-5}}{9.8} = 6.2 \times 10^{-6}\,\text{s} \qquad \text{[oildrop]}.$$

After falling for just 20 microseconds, this oildrop will have acquired 95% of its terminal speed. For almost every purpose, the oildrop *always* travels at its terminal speed.

For the drop of mist of Example 2.2, the terminal speed was $v_{\rm ter} = 1.3$ m/s and so $\tau = v_{\rm ter}/g \approx 0.13$ s. After about 0.4 s, the drop will have acquired 95% of its terminal speed.

Whether or not our falling object starts from rest, we can find its position y as a function of time by integrating the known form (2.30) of v_y,

$$v_y(t) = v_{\rm ter} + (v_{yo} - v_{\rm ter})e^{-t/\tau}.$$

Assuming that the projectile's initial position is $y = 0$, it immediately follows that

$$\begin{aligned} y(t) &= \int_0^t v_y(t')\,dt' \\ &= v_{\rm ter}t + (v_{yo} - v_{\rm ter})\tau\left(1 - e^{-t/\tau}\right). \end{aligned} \tag{2.35}$$

This equation for $y(t)$ can now be combined with Equation (2.23) for $x(t)$ to give us the orbit of any projectile, moving both horizontally and vertically, in a linear medium.

2.3 Trajectory and Range in a Linear Medium

We saw at the begining of the last section that the equation of motion for a projectile moving in any direction resolves into two separate equations, one for the horizontal and one for the vertical motion [Equations (2.15) and (2.16)]. We have solved each of these separate equations in (2.23) and (2.35), and we can now put these solutions together to give the trajectory of an arbitrary projectile moving in any direction. In this discussion it is marginally more convenient to measure y vertically *upward*, in which case we must reverse the sign of v_{ter}. (Make sure you understand this point.) Thus the two equations of the orbit become

$$\left.\begin{array}{rcl} x(t) & = & v_{xo}\tau\left(1-e^{-t/\tau}\right) \\ y(t) & = & (v_{yo}+v_{\text{ter}})\tau\left(1-e^{-t/\tau}\right)-v_{\text{ter}}t. \end{array}\right\} \tag{2.36}$$

You can eliminate t from these two equations by solving the first for t and then substituting into the second. (See Problem 2.17.) The result is the equation for the trajectory:

$$y=\frac{v_{yo}+v_{\text{ter}}}{v_{xo}}x+v_{\text{ter}}\tau\ln\left(1-\frac{x}{v_{xo}\tau}\right). \tag{2.37}$$

This equation is probably too complicated to be especially illuminating, but I have plotted it as the solid curve in Figure 2.7, with the help of which you can understand some of the features of (2.37). For example, if you look at the second term on the right of (2.37), you will see that as $x \to v_{xo}\tau$ the argument of the log function approaches zero; therefore, the log term and hence y both approach $-\infty$. That is, the trajectory has a vertical asymptote at $x = v_{xo}\tau$, as you can see in the picture. I leave it as an exercise (Problem 2.19) for you to check that if air resistance is switched off (v_{ter} and τ both approach infinity), the trajectory defined by (2.37) does indeed approach the dashed trajectory corresponding to zero air resistance.

Horizontal Range

A standard (and quite interesting) problem in elementary physics courses is to show that the horizontal range R of a projectile (subject to no air resistance of course) is

$$R_{\text{vac}}=\frac{2v_{xo}v_{yo}}{g} \qquad \text{[no air resistance]} \tag{2.38}$$

where R_{vac} stands for the range in a vacuum. Let us see how this result is modified by air resistance.

The range R is the value of x when y as given by (2.37) is zero. Thus R is the solution of the equation

$$\frac{v_{yo}+v_{\text{ter}}}{v_{xo}}R+v_{\text{ter}}\tau\ln\left(1-\frac{R}{v_{xo}\tau}\right)=0. \tag{2.39}$$

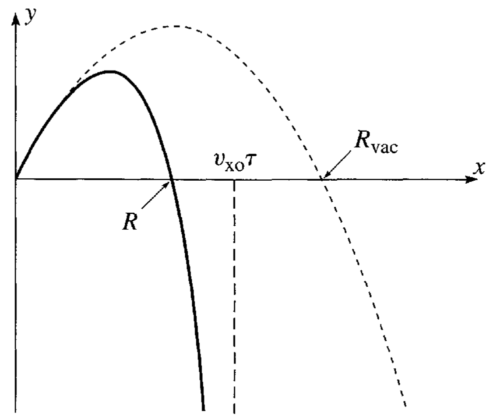

Figure 2.7 The trajectory of a projectile subject to a linear drag force (solid curve) and the corresponding trajectory in a vacuum (dashed curve). At first the two curves are very similar, but as t increases, air resistance slows the projectile and pulls its trajectory down, with a vertical asymptote at $x = v_{xo}\tau$. The horizontal range of the projectile is labeled R, and the corresponding range in vacuum R_{vac}.

This is a transcendental equation and cannot be solved analytically, that is, in terms of well known, elementary functions such as logs, or sines and cosines. For a given choice of parameters, it can be solved numerically with a computer (Problem 2.22), but this approach usually gives one little sense of how the solution depends on the parameters. Often a good alternative is to find some approximation that allows an *approximate* analytic solution. (Before the advent of computers, this was often the only way to find out what happens.) In the present case, it is often clear that the effects of air resistance should be *small*. This means that both v_{ter} and τ are large and the second term in the argument of the log function is small (since it has τ in its denominator). This suggests that we expand the log in a Taylor series (see Problem 2.18):

$$\ln(1-\epsilon) = -\left(\epsilon + \tfrac{1}{2}\epsilon^2 + \tfrac{1}{3}\epsilon^3 + \cdots\right). \qquad (2.40)$$

We can use this expansion for the log term in (2.39), and, provided τ is large enough, we can surely neglect the terms beyond ϵ^3. This gives the equation

$$\left[\frac{v_{yo} + v_{ter}}{v_{xo}}\right] R - v_{ter}\tau \left[\frac{R}{v_{xo}\tau} + \frac{1}{2}\left(\frac{R}{v_{xo}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{xo}\tau}\right)^3\right] = 0. \qquad (2.41)$$

This equation can be quickly tidied up. First, the second term in the first bracket cancels the first term in the second. Next, every term contains a factor of R. This implies that one solution is $R = 0$, which is correct — the height y is zero when $x = 0$. Nevertheless, this is not the solution we are interested in, and we can divide out the common factor of R. A little rearrangement (and replacement of $v_{\rm ter}/\tau$ by g) lets us rewite the equation as

$$R = \frac{2v_{xo}v_{yo}}{g} - \frac{2}{3v_{xo}\tau}R^2. \tag{2.42}$$

This may seem a perverse way to write a quadratic equation for R, but it leads us quickly to the desired approximate solution. The point is that the second term on the right is very small. (In the numerator R is certainly no more than $R_{\rm vac}$ and we are assuming that τ in the denominator is very large.) Therefore, as a first approximation we get

$$R \approx \frac{2v_{xo}v_{yo}}{g} = R_{\rm vac}. \tag{2.43}$$

This is just what we expected: For low air resistance, the range is close to $R_{\rm vac}$. But with the help of (2.42) we can now get a second, better approximation. The last term of (2.42) is the required correction to $R_{\rm vac}$; because it is already small, we would certainly be satisfied with an approximate value for this correction. Thus, in evaluating the last term of (2.42), we can replace R with the approximate value $R \approx R_{\rm vac}$, and we find as our second approximation [remember that the first term in (2.42) is just $R_{\rm vac}$]

$$\begin{aligned} R &\approx R_{\rm vac} - \frac{2}{3v_{xo}\tau}(R_{\rm vac})^2 \\ &= R_{\rm vac}\left(1 - \frac{4}{3}\frac{v_{yo}}{v_{\rm ter}}\right). \end{aligned} \tag{2.44}$$

(To get the second line, I replaced the second $R_{\rm vac}$ in the previous line by $2v_{xo}v_{yo}/g$ and τg by $v_{\rm ter}$.) Notice that the correction for air resistance always makes R smaller than $R_{\rm vac}$, as one would expect. Notice also that the correction depends only on the ratio $v_{yo}/v_{\rm ter}$. More generally, it is easy to see (Problem 2.32) that the importance of air resistance is indicated by the ratio $v/v_{\rm ter}$ of the projectile's speed to the terminal speed. If $v/v_{\rm ter} \ll 1$ throughout the flight, the effect of air resistance is very small; if $v/v_{\rm ter}$ is around 1 or more, air resistance is almost certainly important [and the approximation (2.44) is certainly no good].

EXAMPLE 2.4 Range of Small Metal Pellets

I flick a tiny metal pellet with diameter $d = 0.2$ mm and $\mathbf{v} = 1$ m/s at 45°. Find its horizontal range assuming the pellet is gold (density $\varrho \approx 16\,\text{g/cm}^3$). What if it is aluminum (density $\varrho \approx 2.7\,\text{g/cm}^3$)?

In the absence of air resistance, both pellets would have the same range,

$$R_{\rm vac} = \frac{2v_{xo}v_{yo}}{g} = 10.2\,\text{cm}.$$

For gold, Equation (2.27) gives (as you can check) $v_{\text{ter}} \approx 21$ m/s. Thus the correction term in (2.44) is

$$\frac{4}{3}\frac{v_{yo}}{v_{\text{ter}}} = \frac{4}{3} \times \frac{0.71}{21} \approx 0.05.$$

That is, air resistance reduces the range by 5% to about 9.7 cm. The density of aluminum is about 1/6 times that of gold. Therefore the terminal speed is one sixth as big, and the correction for aluminum is 6 times greater or about 30%, giving a range of about 7 cm. For the gold pellet the correction for air resistance is quite small and could perhaps be neglected; for the aluminum pellet, the correction is still small, but is certainly not negligible.

2.4 Quadratic Air Resistance

In the last two sections we have developed a rather complete theory of projectiles subject to a linear drag force, $\mathbf{f} = -b\mathbf{v}$. While we *can* find examples of projectiles for which the drag is linear (notably very small objects, such as the Millikan oildrop), for most of the more obvious examples of projectiles (baseballs, footballs, cannonballs, and the like) it is a far better approximation to say that the drag is pure quadratic, $\mathbf{f} = -cv^2\hat{\mathbf{v}}$. We must, therefore, develop a corresponding theory for a quadratic drag force. On the face of it, the two theories are not so very different. In either case we have to solve the differential equation

$$m\dot{\mathbf{v}} = m\mathbf{g} + \mathbf{f}, \tag{2.45}$$

and in both cases this is a first-order differential equation for the velocity $\mathbf{v}$, with $\mathbf{f}$ depending in a relatively simple way on $\mathbf{v}$. There is, however, an important difference. In the linear case ($\mathbf{f} = -b\mathbf{v}$), Equation (2.45) is a *linear* differential equation, inasmuch as the terms that involve $\mathbf{v}$ are all linear in $\mathbf{v}$ or its derivatives. In the quadratic case, Equation (2.45) is, of course, nonlinear. And it turns out that the mathematical theory of nonlinear differential equations is significantly more complicated than the linear theory. As a practical matter, we shall find that for the case of a general projectile, moving in both the x and y directions, Equation (2.45) cannot be solved in terms of elementary functions when the drag is quadratic. More generally, we shall see in Chapter 12 that for more complicated systems, nonlinearity can lead to the astonishing phenomenon of chaos, although this does not happen in the present case.

In this section, I shall start with the same two special cases discussed in Section 2.2, a body that is constrained to move horizontally, such as a railroad car on a horizontal track, and a body that moves vertically, such as a stone dropped from a window (both now with quadratic drag forces). We shall find that in these two especially simple cases the differential equation (2.45) *can* be solved by elementary means, and the solutions introduce some important techniques and interesting results. I shall then discuss briefly the general case (motion in both the horizontal and vertical directions), which can be solved only numerically.

Horizontal Motion with Quadratic Drag

Let us consider a body moving horizontally (in the positive x direction), subject to a quadratic drag and no other forces. For example, you could imagine a cycle racer, who has crossed the finishing line and is coasting to a stop under the influence of air resistance. To the extent that the cycle is well lubricated and tires well inflated, we can ignore ordinary friction,[3] and, except at very low speeds, air resistance is purely quadratic. The x component of the equation of motion is therefore (I'll abbreviate v_x to v)

$$m\frac{dv}{dt} = -cv^2. \tag{2.46}$$

If we divide by v^2 and multiply by dt, we get an equation in which only the variable v appears on the left and only t on the right:[4]

$$m\frac{dv}{v^2} = -c\,dt. \tag{2.47}$$

This trick — of rearranging a differential equation so that only one variable appears on the left and only the other on the right — is called **separation of variables**. When it is possible, separation of variables is often the simplest way to solve a first-order differential equation, since the solution can be found by simple integration of both sides.

Integrating Equation (2.47) we find

$$m\int_{v_o}^{v}\frac{dv'}{v'^2} = -c\int_0^t dt'$$

where v_o is the initial velocity at $t = 0$. Notice that I have written both sides as definite integrals, with the appropriate limits, so that I shan't have to worry about any constants of integration. I have also renamed the variables of integration as v' and t' to avoid

[3] As I shall discuss shortly, when the cyclist slows down to a stop, air resistance becomes smaller, and eventually friction becomes the dominant force. Nevertheless, at speeds around 10 mph or more, it is a fair approximation to ignore everything but the quadratic air resistance.

[4] In passing from (2.46) to (2.47), I have treated the derivative dv/dt as if it were the quotient of two separate numbers, dv and dt. As you are certainly aware this cavalier proceeding is not strictly correct. Nevertheless, it can be justified in two ways. First, in the theory of *differentials*, it is in fact true that dv and dt are defined as separate numbers (differentials), such that their quotient is the derivative dv/dt. Fortunately, it is quite unnecessary to know about this theory. As physicists we know that dv/dt is the limit of $\Delta v/\Delta t$, as both Δv and Δt become small, and I shall take the view that dv is just shorthand for Δv (and likewise dt for Δt), *with the understanding that it has been taken small enough that the quotient dv/dt is within my desired accuracy of the true derivative.* With this understanding, (2.47), with dv on one side and dt on the other, makes perfectly good sense.

confusion with the upper limits v and t. Both of these integrals are easily evaluated, and we find

$$m\left(\frac{1}{v_o}-\frac{1}{v}\right)=-ct \tag{2.48}$$

or, solving for v,

$$v(t)=\frac{v_o}{1+cv_ot/m}=\frac{v_o}{1+t/\tau} \tag{2.49}$$

where I have introduced the abbreviation τ for the combination of constants

$$\tau=\frac{m}{cv_o} \qquad \text{[for quadratic drag].} \tag{2.50}$$

As you can easily check, τ is a time, with the significance that when $t=\tau$ the velocity is $v=v_o/2$. Notice that this parameter τ is different from the τ introduced in (2.22) for motion subject to linear air resistance; nevertheless, both parameters have the same general significance as indicators of the time for air resistance to slow the motion appreciably.

To find the bicycle's position x, we have only to integrate v to give (as you should check)

$$\begin{aligned} x(t) &= x_o+\int_0^t v(t')\,dt' \\ &= v_o\tau\ln(1+t/\tau), \end{aligned} \tag{2.51}$$

if we take the initial position x_o to be zero. Figure 2.8 shows our results for v and x as functions of t. It is interesting to compare these graphs with the corresponding graphs of Figure 2.4 for a body coasting horizontally but subject to a linear resistance. Superficially, the two graphs for the velocity look similar. In particular, both go to zero as $t\to\infty$. But in the linear case v goes to zero *exponentially*, whereas in the quadratic case it does so only very slowly, like $1/t$. This difference in the behavior of v manifests itself quite dramatically in the behavior of x. In the linear case, we

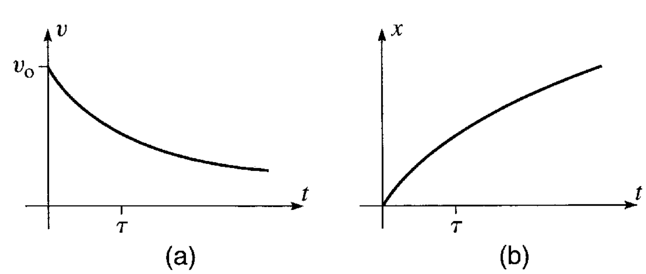

Figure 2.8 The motion of a body, such as a bicycle, coasting horizontally and subject to a quadratic air resistance. **(a)** The velocity is given by (2.49) and goes to zero like $1/t$ as $t\to\infty$. **(b)** The position is given by (2.51) and goes to infinity as $t\to\infty$.

saw that x approaches a finite limit as $t \to \infty$, but it is clear from (2.51) that in the quadratic case x increases without limit as $t \to \infty$.

The striking difference in the behavior of x for quadratic and linear drags is easy to understand qualitatively. In the quadratic case, the drag is proportional to v^2. Thus as v gets small, the drag gets *very* small — so small that it fails to bring the bicyle to rest at any finite value of x. This unexpected behavior serves to highlight that a drag force that is proportional to v^2 at *all* speeds is unrealistic. Although the linear drag and ordinary friction are very small, nevertheless as $v \to 0$ they must eventually become more important than the v^2 term and cannot be ignored. In particular, one or another of these two terms (friction in the case of a bicycle) ensures that no real body can coast on to infinity!

Vertical Motion with Quadratic Drag

The case that an object moves vertically with a quadratic drag force can be solved in much the same way as the horizontal case. Consider a baseball that is dropped from a window in a high tower. If we measure the coordinate y vertically down, the equation of motion is (I'll abbreviate v_y to v now)

$$m\dot{v} = mg - cv^2. \tag{2.52}$$

Before we solve this equation, let us consider the ball's terminal speed, the speed at which the two terms on the right of (2.52) just balance. Evidently this must satisfy $cv^2 = mg$, whose solution is

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}. \tag{2.53}$$

For any given object (given m, g, and c), this lets us calculate the terminal speed. For example, for a baseball it gives (as we shall see in a moment) $v_{\text{ter}} \approx 35$ m/s, or nearly 80 miles per hour.

We can tidy the equation of motion (2.52) a little by using (2.53) to replace c by mg/v_{ter}^2 and canceling the factors of m:

$$\dot{v} = g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right). \tag{2.54}$$

This can be solved by separation of variables, just as in the case of horizontal motion: First we can rewrite it as

$$\frac{dv}{1 - v^2/v_{\text{ter}}^2} = g\,dt. \tag{2.55}$$

This is the desired separated form (only v on the left and only t on the right) and we can simply integrate both sides.[5] Assuming the ball starts from rest, the limits of

[5]Notice that in fact any one-dimensional problem where the net force depends only on the velocity can be solved by separation of variables, since the equation $m\dot{v} = F(v)$ can always be

integration are 0 and v on the left and 0 and t on the right, and we find (as you should verify — Problem 2.35)

$$\frac{v_{\text{ter}}}{g} \operatorname{arctanh}\left(\frac{v}{v_{\text{ter}}}\right) = t \tag{2.56}$$

where "arctanh" denotes the inverse hyperbolic tangent. This particular integral can be evaluated alternatively in terms of the natural log function (Problem 2.37). However, the hyperbolic functions, sinh, cosh, and tanh, and their inverses arcsinh, arccosh, and arctanh, come up so often in all branches of physics that you really should learn to use them. If you have not had much exposure to them, you might want to look at Problems 2.33 and 2.34, and study graphs of these functions.

Equation (2.56) can be solved for v to give

$$v = v_{\text{ter}} \tanh\left(\frac{gt}{v_{\text{ter}}}\right). \tag{2.57}$$

To find the position y, we just integrate v to give

$$y = \frac{(v_{\text{ter}})^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]. \tag{2.58}$$

While both of these two formulas can be cleaned up a little (see Problem 2.35), they are already sufficient to work the following example.

EXAMPLE 2.5 A Baseball Dropped from a High Tower

Find the terminal speed of a baseball (mass $m = 0.15$ kg and diameter $D = 7$ cm). Make plots of its velocity and position for the first six seconds after it is dropped from a tall tower.

The terminal speed is given by (2.53), with the coefficient of air resistance c given by (2.4) as $c = \gamma D^2$ where $\gamma = 0.25\ \text{N·s}^2/\text{m}^4$. Therefore

$$v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{(0.15\,\text{kg}) \times (9.8\,\text{m/s}^2)}{(0.25\,\text{N·s}^2/\text{m}^4) \times (0.07\,\text{m})^2}} = 35\ \text{m/s} \tag{2.59}$$

or nearly 80 miles per hour. It is interesting to note that fast baseball pitchers can pitch a ball considerably *faster* than v_{ter}. Under these conditions, the drag force is actually *greater* than the ball's weight!

The plots of v and y can be made by hand, but are, of course, much easier with the help of computer software such as Mathcad or Mathematica that can make the plots for you. Whatever method we choose, the results are as shown in Figure 2.9, where the solid curves show the actual velocity and position while the dashed curves are the corresponding values in a vacuum. The actual velocity levels out,

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written as $m\,dv/F(v) = dt$. Of course there is no assurance that this can be integrated analytically if $F(v)$ is too complicated, but it does guarantee a straightforward numerical solution at worst. See Problem 2.7.

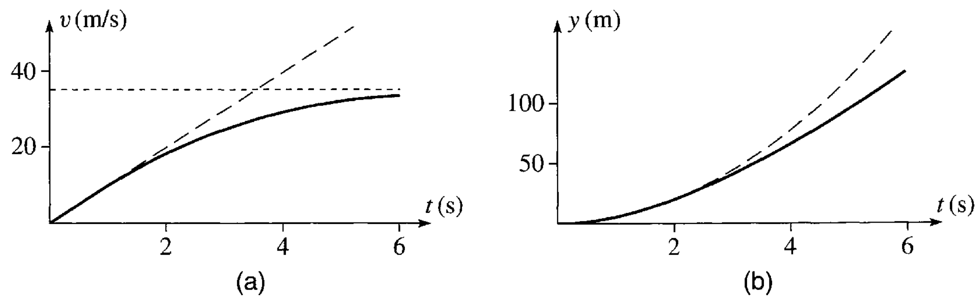

Figure 2.9 The motion of a baseball dropped from the top of a high tower (solid curves). The corresponding motion in a vacuum is shown with long dashes. **(a)** The actual velocity approaches the ball's terminal velocity $v_{\text{ter}} = 35$ m/s as $t \to \infty$. **(b)** The graph of position against time falls further and further behind the corresponding vacuum graph. When $t = 6$ s, the baseball has dropped about 130 meters; in a vacuum, it would have dropped about 180 meters.

approaching the terminal value $v_{\text{ter}} = 35$ m/s as $t \to \infty$, whereas the velocity in a vacuum would increase without limit. Initially, the position increases just as it would in a vacuum (that is, $y = \frac{1}{2}gt^2$), but falls behind as v increases and the air resistance becomes more important. Eventually, y approaches a straight line of the form $y = v_{\text{ter}}t+$ const. (See Problem 2.35.)

Quadratic Drag with Horizontal and Vertical Motion

The equation of motion for a projectile subject to quadratic drag,

$$\begin{aligned} m\ddot{\mathbf{r}} &= m\mathbf{g} - cv^2\hat{\mathbf{v}} \\ &= m\mathbf{g} - cv\mathbf{v}, \end{aligned} \tag{2.60}$$

resolves into its horizontal and vertical components (with y measured vertically upward) to give

$$\left.\begin{aligned} m\dot{v}_x &= -c\sqrt{v_x^2 + v_y^2}\, v_x \\ m\dot{v}_y &= -mg - c\sqrt{v_x^2 + v_y^2}\, v_y. \end{aligned}\right\} \tag{2.61}$$

These are two differential equations for the two unknown functions $v_x(t)$ and $v_y(t)$, but each equation involves *both* v_x and v_y. In particular, neither equation is the same as for an object that moves only in the x direction or only in the y direction. This means that we cannot solve these two equations by simply pasting together our two separate solutions for horizontal and vertical motion. Worse still, it turns out that the two equations (2.61) cannot be solved analytically at all. The only way to solve them is numerically, which we can only do for specified numerical initial conditions (that is, specified values of the initial position and velocity). This means that we cannot find the *general* solution; all we can do numerically is to find the particular solution corresponding to any chosen initial conditions. Before I discuss some general properties of the solutions of (2.61), let us work out one such numerical solution.

EXAMPLE 2.6 Trajectory of a Baseball

The baseball of Example 2.5 is now thrown with velocity 30 m/s (about 70 mi/h) at 50° above the horizontal from a high cliff. Find its trajectory for the first eight seconds of flight and compare with the corresponding trajectory in a vacuum. If the same baseball was thrown with the same initial velocity on horizontal ground how far would it travel before landing? That is, what is its horizontal range?

We have to solve the two coupled differential equations (2.61) with the initial conditions

$$v_{xo} = v_o \cos\theta = 19.3 \text{ m/s} \quad \text{and} \quad v_{yo} = v_o \sin\theta = 23.0 \text{ m/s}$$

and $x_o = y_o = 0$ (if we put the origin at the point from which the ball is thrown). This can be done with systems such as Mathematica, Matlab, or Maple, or with programming languages such as "C" or Fortran. Figure 2.10 shows the resulting trajectory, found using the function "NDSolve" in Mathematica.

Several features of Figure 2.10 deserve comment. Obviously the effect of air resistance is to lower the trajectory, as compared to the vacuum trajectory (shown dashed). For example, we see that in a vacuum the high point of the trajectory occurs at $t \approx 2.3$ s and is about 27 m above the starting point; with air resistance, the high point comes just before $t = 2.0$ s and is at about 21 m. In a vacuum, the ball would continue to move indefinitely in the x direction. The

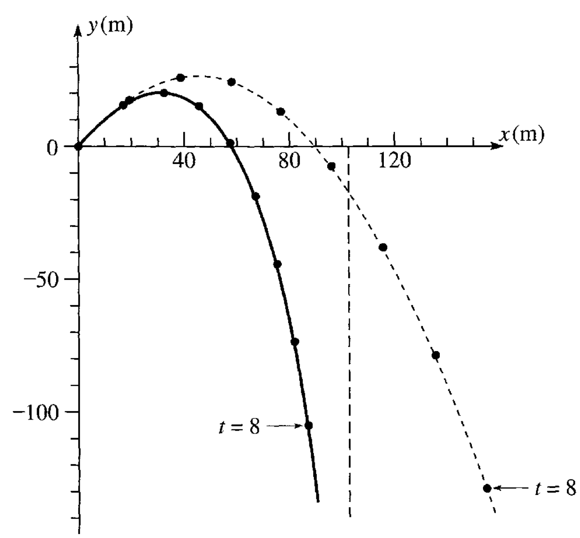

Figure 2.10 Trajectory of a baseball thrown off a cliff and subject to quadratic air resistance (solid curve). The initial velocity is 30 m/s at 50° above the horizontal; the terminal speed is 35 m/s. The dashed curve shows the corresponding trajectory in a vacuum. The dots show the ball's position at one-second intervals. Air resistance slows the horizontal motion, so that the ball approaches a vertical asymptote just beyond $x = 100$ meters.

effect of air resistance is to slow the horizontal motion so that x never moves to the right of a vertical asymptote near $x = 100$ m.

The horizontal range of the baseball is easily read off the figure as the value of x when y returns to zero. We see that $R \approx 59$ m, as opposed to the range in vacuum, $R_{vac} \approx 90$ m. The effect of air resistance is quite large in this example, as we might have anticipated: The ball was thrown with a speed only a little less than the terminal speed (30 vs 35 m/s), and this means that the force of air resistance is only a little less than that of gravity. This being the case, we should expect air resistance to change the trajectory appreciably.

This example illustrates several of the general features of projectile motion with a quadratic drag force. Although we cannot solve analytically the equations of motion (2.61) for this problem, we *can* use the equations to prove various general properties of the trajectory. For example, we noticed that the baseball reached a lower maximum height, and did so sooner, than it would have in a vacuum. It is easy to prove that this will always be the case: As long as the projectile is moving upward ($v_y > 0$), the force of air resistance has a *downward* y component. Thus the downward acceleration is greater than g (its value in vacuum). Therefore a graph of v_y against t slopes down from v_{yo} more quickly than it would in vacuum, as shown in Figure 2.11. This guarantees that v_y reaches zero sooner than it would in vacuum, and that the ball travels less distance (in the y direction) before reaching the high point. That is, the ball's high point occurs sooner, and is lower, than it would be in a vacuum.

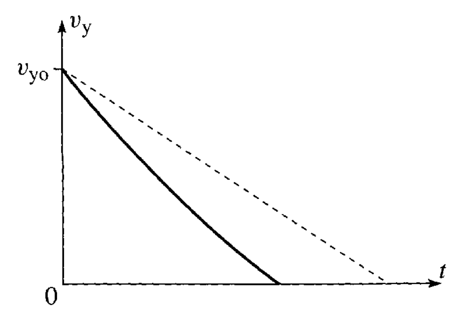

Figure 2.11 Graph of v_y against t for a projectile that is thrown upward ($v_{yo} > 0$) and is subject to a quadratic resistance (solid curve). The dashed line (slope $= -g$) is the corresponding graph when there is no air resistance. The projectile moves upward until it reaches its maximum height when $v_y = 0$. During this time, the drag force is downward and the downward acceleration is always greater than g. Therefore, the curve slopes more steeply than the dashed line, and the projectile reaches its high point sooner than it would in a vacuum. Since the area under the curve is less than that under the dashed line, the projectile's maximum height is less than it would be in a vacuum.

I claimed that the baseball of Example 2.6 approaches a vertical asymptote as $t \to \infty$, and we can now prove that this is always the case. First, it is easy to convince yourself that once the ball starts moving downward, it continues to accelerate downward, with v_y approaching $-v_{\text{ter}}$ as $t \to \infty$. At the same time v_x continues to decrease and approaches zero. Thus the square root in both of the equations (2.61) approaches v_{ter}. In particular, when t is large, the equation for v_x can be approximated by

$$\dot{v}_x \approx -\frac{cv_{\text{ter}}}{m} v_x = -kv_x$$

say. The solution of this equation is, of course, an exponential function, $v_x = Ae^{-kt}$, and we see that v_x approaches zero very rapidly (exponentially) as $t \to \infty$. This guarantees that x, which is the integral of v_x,

$$x(t) = \int_0^t v_x(t')\, dt',$$

approaches a finite limit as $t \to \infty$, and the trajectory has a finite vertical asymptote as claimed.

2.5 Motion of a Charge in a Uniform Magnetic Field

Another interesting application of Newton's laws, and (like projectile motion) an application that lets me introduce some important mathematical methods, is the motion of a charged particle in a magnetic field. I shall consider here a particle of charge q (which I shall usually take to be positive), moving in a uniform magnetic field $\mathbf{B}$ that points in the z direction as shown in Figure 2.12. The net force on the particle is just the magnetic force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}, \tag{2.62}$$

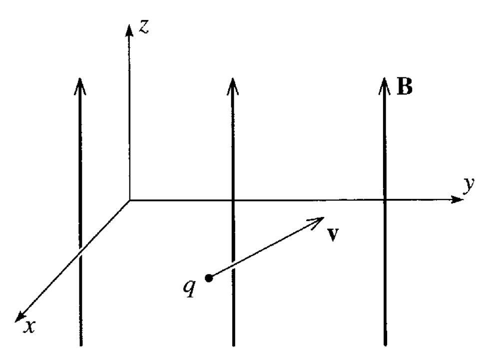

Figure 2.12 A charged particle moving in a uniform magnetic field that points in the z direction.

so the equation of motion can be written as

$$m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B}. \tag{2.63}$$

[As with projectiles, the force depends only on the velocity (not the position), so the second law reduces to a first-order differential equation for $\mathbf{v}$.]

As is so often the case, the simplest way to solve the equation of motion is to resolve it into components. The components of $\mathbf{v}$ and $\mathbf{B}$ are

$$\mathbf{v} = (v_x, v_y, v_z)$$

and

$$\mathbf{B} = (0, 0, B),$$

from which we can read off the components of $\mathbf{v} \times \mathbf{B}$:

$$\mathbf{v} \times \mathbf{B} = (v_y B, -v_x B, 0).$$

Thus the three components of (2.63) are

$$m\dot{v}_x = qBv_y \tag{2.64}$$

$$m\dot{v}_y = -qBv_x \tag{2.65}$$

$$m\dot{v}_z = 0. \tag{2.66}$$

The last of these says simply that v_z, the component of the particle's velocity in the direction of $\mathbf{B}$, is constant:

$$v_z = \text{const},$$

a result we could have anticipated since the magnetic force is always perpendicular to $\mathbf{B}$. Because v_z is constant, we shall focus most of our attention on v_x and v_y. In fact, we can even think of them as comprising a two-dimensional vector (v_x, v_y), which is just the projection of $\mathbf{v}$ onto the xy plane and can be called the *transverse velocity*,

$$(v_x, v_y) = \text{transverse velocity}.$$

To simplify the equations (2.64) and (2.65) for v_x and v_y, I shall define the parameter

$$\omega = \frac{qB}{m}, \tag{2.67}$$

which has the dimensions of inverse time and is called the **cyclotron frequency**. With this notation, Equations (2.64) and (2.65) become

$$\left.\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x. \end{aligned}\right\} \tag{2.68}$$

These two coupled differential equations can be solved in a host of different ways. I would like to describe one that makes use of complex numbers. Though perhaps

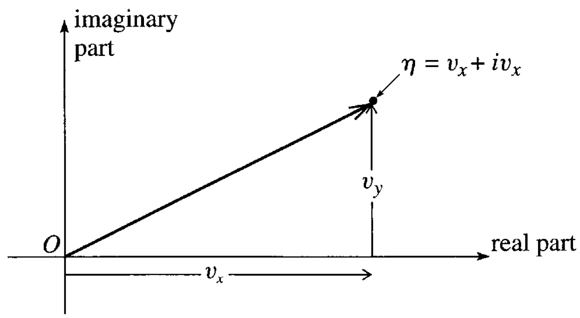

Figure 2.13 The complex number $\eta = v_x + iv_y$ is represented as a point in the complex plane. The arrow pointing from O to η is literally a picture of the transverse velocity vector (v_x, v_y).

not the easiest solution, this method has surprisingly wide application in many areas of physics. (For an alternative solution that avoids complex numbers, see Problem 2.54.)

The two variables v_x and v_y are, of course, real numbers. However, there is nothing to prevent us from defining a complex number

$$\eta = v_x + iv_y, \tag{2.69}$$

where i denotes the square root of -1 (called j by most engineers), $i = \sqrt{-1}$ (and η is the Greek letter eta). If we draw the complex number η in the complex plane, or Argand diagram, then its two components are v_x and v_y as shown in Figure 2.13; in other words, the representation of η in the complex plane is a picture of the two-dimensional transverse velocity (v_x, v_y).

The advantages of introducing the complex number η appear when we evaluate its derivative. Using (2.68), we find that

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - i\omega v_x = -i\omega(v_x + iv_y)$$

or

$$\dot{\eta} = -i\omega\eta. \tag{2.70}$$

We see that the two coupled equations for v_x and v_y have become a single equation for the complex number η. Furthermore, it is an equation of the now familiar form $\dot{u} = ku$, whose solution we know to be the exponential $u = Ae^{kt}$. Thus we can immediately write down the solution for η:

$$\eta = Ae^{-i\omega t} \tag{2.71}$$

Before we discuss the significance of this solution, I would like to review a few properties of complex exponentials in the next section. If you are very familiar with these ideas, by all means skip this material.

2.6 Complex Exponentials

While you are certainly familiar with the exponential function e^x for a real variable x, you may not be so at home with e^z when z is complex.[6] For the real case there are several possible definitions of e^x (for instance, as the function that is equal to its own derivative). The definition that extends most easily to the complex case is the Taylor series (see Problem 2.18)

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \tag{2.72}$$

For any value of z, real or complex, large or small, this series converges to give a well-defined value for e^z. By differentiating it, you can easily convince yourself that it has the expected property that it equals its own derivative. And one can show (not always so easily) that it has all the other familiar properties of the exponential function — for instance, that $e^z e^w = e^{(z+w)}$. (See Problems 2.50 and 2.51.) In particular, the function Ae^{kz} (with A and k any constants, real or complex) has the property that

$$\frac{d}{dz}\left(Ae^{kz}\right) = k\left(Ae^{kz}\right). \tag{2.73}$$

Since it satisfies this same equation whatever the value of A, it is, in fact, the general solution of the first-order equation $df/dz = kf$. At the end of the last section, I introduced the complex number $\eta(t)$ and showed that it satisfied the equation $\dot{\eta} = -i\omega\eta$. We are now justified in saying that this guarantees that η must be the exponential function anticipated in (2.71).

We shall be particularly concerned with the exponential of a pure imaginary number, that is, $e^{i\theta}$ where θ is a real number. The Taylor series (2.72) for this function reads

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots. \tag{2.74}$$

Noting that $i^2 = -1$, $i^3 = -i$, and so on, you can see that all of the even powers in this series are real, while all of the odd powers are pure imaginary. Regrouping accordingly, we can rewrite (2.74) to read

$$e^{i\theta} = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \cdots\right]. \tag{2.75}$$

The series in the first brackets is the Taylor series for $\cos\theta$, and that in the second brackets is $\sin\theta$ (Problem 2.18). Thus we have proved the important relation:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.76}$$

[6] For a review of some elementary properties of complex numbers, see Problems 2.45 to 2.49.

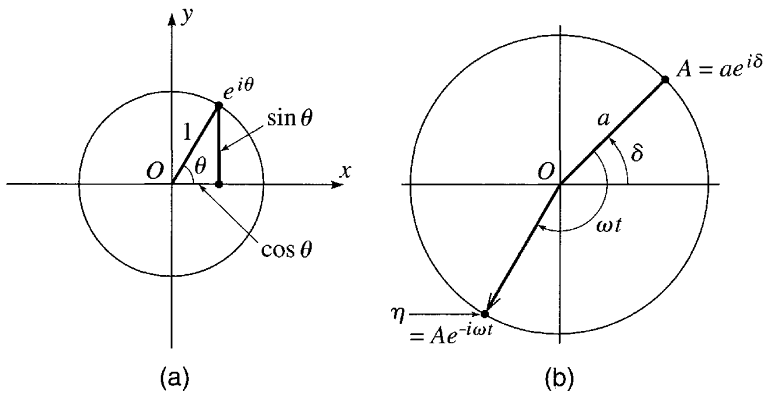

Figure 2.14 **(a)** Euler's formula, (2.76), implies that the complex number $e^{i\theta}$ lies on the unit circle (the circle of radius 1, centered on the origin O) with polar angle θ. **(b)** The complex constant $A = ae^{i\delta}$ lies on a circle of radius a with polar angle δ. The function $\eta(t) = Ae^{-i\omega t}$ lies on the same circle but with polar angle $(\delta - \omega t)$ and moves clockwise around the circle as t advances.

This result, known as **Euler's formula**, is illustrated in Figure 2.14(a). Note especially that the complex number $e^{i\theta}$ has polar angle θ and, since $\cos^2\theta + \sin^2\theta = 1$, the magnitude of $e^{i\theta}$ is 1; that is, $e^{i\theta}$ lies on the *unit circle*, the circle with radius 1 centered at O.

Our main concern is with a complex number of the form $\eta = Ae^{-i\omega t}$. The coefficient A is a fixed complex number, which can be expressed as $A = ae^{i\delta}$, where $a = |A|$ is the magnitude, and δ is the polar angle of A, as shown in Figure 2.14(b). (See Problem 2.45.) The number η can therefore be written as

$$\eta = Ae^{-i\omega t} = ae^{i\delta}e^{-i\omega t} = ae^{i(\delta-\omega t)}. \tag{2.77}$$

Thus η has the same magnitude as A (namely a), but has polar angle equal to $(\delta - \omega t)$, as shown in Figure 2.14(b). As a function of t, the number η moves clockwise around the circle of radius a with angular velocity ω.

It is important that you get a good feel for the role of the complex constant $A = ae^{i\delta}$ in (2.77): If A happened to equal 1, then η would be just $\eta = e^{-i\omega t}$, which lies on the unit circle, moving clockwise with angular velocity ω and starting from the real axis ($\eta = 1$) when $t = 0$. If $A = a$ is real but not equal to 1, then it simply magnifies the unit circle to a circle of radius a, around which η moves with the same angular speed and starting from the real axis, at $\eta = a$ when $t = 0$. Finally if $A = ae^{i\delta}$, then the effect of the angle δ is to rotate η through the fixed angle δ, so that η starts out at $t = 0$ with polar angle δ.

Armed with these mathematical results, we can now return to the charged particle in a magnetic field.

2.7 Solution for the Charge in a B Field

Mathematically, the solution for the velocity $\mathbf{v}$ of our charged particle in a B field is complete, and all that remains is to interpret it physically. We already know that v_z, the component along $\mathbf{B}$, is constant. The components (v_x, v_y) transverse to $\mathbf{B}$ we have represented by the complex number $\eta = v_x + iv_y$, and we have seen that Newton's second law implies that η has the time dependence $\eta = Ae^{-i\omega t}$, moving uniformly around the circle of Figure 2.14(b). Now, the arrow shown in that figure, pointing from O to η, is in fact a pictorial representation of the transverse velocity (v_x, v_y). Therefore this transverse velocity changes direction, turning clockwise, with constant angular velocity[7] $\omega = qB/m$ and with constant magnitude. Because v_z is constant, this suggests that the particle undergoes a spiralling, or helical, motion. To verify this, we have only to integrate $\mathbf{v}$ to find $\mathbf{r}$ as a function of t.

That v_z is constant implies that

$$z(t) = z_o + v_{zo}t. \tag{2.78}$$

The motion of x and y is most easily found by introducing another complex number

$$\xi = x + iy$$

where ξ is the Greek letter xi. In the complex plane, ξ is a picture of the transverse position (x, y). Clearly, the derivative of ξ is η, that is, $\dot{\xi} = \eta$. Therefore,

$$\begin{aligned} \xi &= \int \eta\, dt = \int Ae^{-i\omega t}\, dt \\ &= \frac{iA}{\omega}e^{-i\omega t} + \text{constant}. \end{aligned} \tag{2.79}$$

If we rename the coefficient iA/ω as C and the constant of integration as $X + iY$, this implies that

$$x + iy = Ce^{-i\omega t} + (X + iY).$$

By redefining our origin so that the z axis goes through the point (X, Y), we can eliminate the constant term on the right to give

$$x + iy = Ce^{-i\omega t}, \tag{2.80}$$

and, by setting $t = 0$, we can identify the remaining constant C as

$$C = x_o + iy_o.$$

This result is illustrated in Figure 2.15. We see there that the transverse position (x, y) moves clockwise round a circle with angular velocity $\omega = qB/m$. Meanwhile z as given by (2.78) increases steadily, so the particle actually describes a uniform helix whose axis is parallel to the magnetic field.

[7] I am assuming the charge q is positive; if q is negative, then $\omega = qB/m$ is negative, meaning that the transverse velocity rotates counterclockwise.

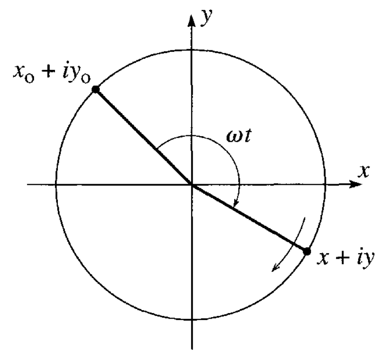

Figure 2.15 Motion of a charge in a uniform magnetic field in the z direction. The transverse position (x, y) moves around a circle as shown, while the coordinate z moves with constant velocity into or out of the page.

There are many examples of the helical motion of a charged particle along a magnetic field; for example, cosmic-ray particles (charged particles hitting the earth from space) can get caught by the earth's magnetic field and spiral north or south along the field lines. If the z component of the velocity happens to be zero, then the spiral reduces to a circle. In the cyclotron, a device for accelerating charged particles to high energies, the particles are trapped in circular orbits in this way. They are slowly accelerated by the judiciously timed application of an electric field. The angular frequency of the orbit is, of course, $\omega = qB/m$ (which is why this is called the cyclotron frequency). The radius of the orbit is

$$r = \frac{v}{\omega} = \frac{mv}{qB} = \frac{p}{qB}. \tag{2.81}$$

This radius increases as the particles accelerate, so that they eventually emerge at the outer edge of the circular magnets that produce the magnetic field.

The same method that we have used here for a charge in a magnetic field can also be used for a particle in magnetic *and electric* fields, but since this complication adds nothing to the method of solution, I shall leave you to try it for yourself in Problems 2.53 and 2.55.

Principal Definitions and Equations of Chapter 2

Linear and Quadratic Drags

Provided the speed v is well below that of sound, the magnitude of the drag force $\mathbf{f} = -f(v)\hat{\mathbf{v}}$ on an object moving through a fluid is usually well approximated as

$$f(v) = f_{\text{lin}} + f_{\text{quad}}$$

where

$$f_{\text{lin}} = bv = \beta D v \quad \text{and} \quad f_{\text{quad}} = cv^2 = \gamma D^2 v^2. \quad \text{[Eqs. (2.2) to (2.6)]}$$

Here D denotes the linear size of the object. For a sphere, D is the diameter and, for a sphere in air at STP, $\beta = 1.6 \times 10^{-4}$ N·s/m^2 and $\gamma = 0.25$ N·s^2/m^4.

The Lorentz Force on a Charged Particle

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \qquad \text{[Eq. (2.62) \& Problem 2.53]}$$

Problems for Chapter 2

Stars indicate the approximate level of difficulty, from easiest (★) to most difficult (★★★).

SECTION 2.1 Air Resistance

2.1 ★ When a baseball flies through the air, the ratio $f_{\text{quad}}/f_{\text{lin}}$ of the quadratic to the linear drag force is given by (2.7). Given that a baseball has diameter 7 cm, find the approximate speed v at which the two drag forces are equally important. For what approximate range of speeds is it safe to treat the drag force as purely quadratic? Under normal conditions is it a good approximation to ignore the linear term? Answer the same questions for a beach ball of diameter 70 cm.

2.2 ★ The origin of the linear drag force on a sphere in a fluid is the viscosity of the fluid. According to Stokes's law, the viscous drag on a sphere is

$$f_{\text{lin}} = 3\pi\eta D v \tag{2.82}$$

where η is the viscosity[8] of the fluid, D the sphere's diameter, and v its speed. Show that this expression reproduces the form (2.3) for f_{lin}, with b given by (2.4) as $b = \beta D$. Given that the viscosity of air at STP is $\eta = 1.7 \times 10^{-5}$ N·s/m^2, verify the value of β given in (2.5).

2.3 ★ **(a)** The quadratic and linear drag forces on a moving sphere in a fluid are given by (2.84) and (2.82) (Problems 2.4 and 2.2). Show that the ratio of these two kinds of drag force can be written as $f_{\text{quad}}/f_{\text{lin}} = R/48$,[9] where the dimensionless **Reynolds number** R is

$$R = \frac{Dv\varrho}{\eta} \tag{2.83}$$

where D is the sphere's diameter, v its speed, and ϱ and η are the fluid's density and viscosity. Clearly the Reynolds number is a measure of the relative importance of the two kinds of drag.[10] When R is

[8] For the record, the viscosity η of a fluid is defined as follows: Imagine a wide channel along which fluid is flowing (x direction) such that the velocity v is zero at the bottom ($y = 0$) and increases toward the top ($y = h$), so that successive layers of fluid slide across one another with a velocity gradient dv/dy. The force F with which an area A of any one layer drags the fluid above it is proportional to A and to dv/dy, and η is defined as the constant of proportionality; that is, $F = \eta A\, dv/dy$.

[9] The numerical factor 48 is for a sphere. A similar result holds for other bodies, but the numerical factor is different for different shapes.

[10] The Reynolds number is usually defined by (2.83) for flow involving any object, with D defined as a typical linear dimension. One sometimes hears the claim that R is the ratio $f_{\text{quad}}/f_{\text{lin}}$. Since $f_{\text{quad}}/f_{\text{lin}} = R/48$ for a sphere, this claim would be better phrased as "R is roughly of the order of $f_{\text{quad}}/f_{\text{lin}}$."

very large, the quadratic drag is dominant and the linear can be neglected; vice versa when R is very small. **(b)** Find the Reynolds number for a steel ball bearing (diameter 2 mm) moving at 5 cm/s through glycerin (density 1.3 g/cm^3 and viscosity 12 N·s/m^2 at STP).

2.4 ★★ The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. **(a)** Assuming the projectile has a cross-sectional area A (normal to its velocity) and speed v, and that the density of the fluid is ϱ, show that the rate at which the projectile encounters fluid (mass/time) is $\varrho A v$. **(b)** Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is $\varrho A v^2$. It is certainly not true that all the fluid that the projectile encounters is accelerated to the full speed v, but one might guess that the actual force would have the form

$$f_{\text{quad}} = \kappa \varrho A v^2 \tag{2.84}$$

where κ is a number less than 1, which would depend on the shape of the projectile, with κ small for a streamlined body, and larger for a body with a flat front end. This proves to be true, and for a sphere the factor κ is found to be $\kappa = 1/4$. **(c)** Show that (2.84) reproduces the form (2.3) for f_{quad}, with c given by (2.4) as $c = \gamma D^2$. Given that the density of air at STP is $\varrho = 1.29$ kg/m^3 and that $\kappa = 1/4$ for a sphere, verify the value of γ given in (2.6).

SECTION 2.2 Linear Air Resistance

2.5 ★ Suppose that a projectile which is subject to a linear resistive force is thrown vertically down with a speed v_{yo} which is *greater* than the terminal speed v_{ter}. Describe and explain how the velocity varies with time, and make a plot of v_y against t for the case that $v_{yo} = 2v_{\text{ter}}$.

2.6 ★ **(a)** Equation (2.33) gives the velocity of an object dropped from rest. At first, when v_y is small, air resistance should be unimportant and (2.33) should agree with the elementary result $v_y = gt$ for free fall in a vacuum. Prove that this is the case. [*Hint:* Remember the Taylor series for $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, for which the first two or three terms are certainly a good approximation when x is small.] **(b)** The position of the dropped object is given by (2.35) with $v_{yo} = 0$. Show similarly that this reduces to the familiar $y = \frac{1}{2}gt^2$ when t is small.

2.7 ★ There are certain simple one-dimensional problems where the equation of motion (Newton's second law) can always be solved, or at least reduced to the problem of doing an integral. One of these (which we have met a couple of times in this chapter) is the motion of a one-dimensional particle subject to a force that depends only on the velocity v, that is, $F = F(v)$. Write down Newton's second law and separate the variables by rewriting it as $m\,dv/F(v) = dt$. Now integrate both sides of this equation and show that

$$t = m \int_{v_o}^{v} \frac{dv'}{F(v')}.$$

Provided you can do the integral, this gives t as a function of v. You can then solve to give v as a function of t. Use this method to solve the special case that $F(v) = F_o$, a constant, and comment on your result. This method of *separation of variables* is used again in Problems 2.8 and 2.9.

2.8 ★ A mass m has velocity v_o at time $t = 0$ and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the method of Problem 2.7 to find v in terms of the time t and the other given parameters. At what time (if any) will it come to rest?

2.9 ★ We solved the differential equation (2.29), $m\dot{v}_y = -b(v_y - v_{ter})$, for the velocity of an object falling through air, by inspection — a most respectable way of solving differential equations. Nevertheless, one would sometimes like a more systematic method, and here is one. Rewrite the equation in the "separated" form

$$\frac{m\,dv_y}{v_y - v_{ter}} = -b\,dt$$

and integrate both sides from time 0 to t to find v_y as a function of t. Compare with (2.30).

2.10 ★★ For a steel ball bearing (diameter 2 mm and density 7.8 g/cm^3) dropped in glycerin (density 1.3 g/cm^3 and viscosity 12 N·s/m^2 at STP), the dominant drag force is the linear drag given by (2.82) of Problem 2.2. **(a)** Find the characteristic time τ and the terminal speed v_{ter}. [In finding the latter, you should include the buoyant force of Archimedes. This just adds a third force on the right side of Equation (2.25).] How long after it is dropped from rest will the ball bearing have reached 95% of its terminal speed? **(b)** Use (2.82) and (2.84) (with $\kappa = 1/4$ since the ball bearing is a sphere) to compute the ratio f_{quad}/f_{lin} at the terminal speed. Was it a good approximation to neglect f_{quad}?

2.11 ★★ Consider an object that is thrown vertically up with initial speed v_o in a linear medium. **(a)** Measuring y *upward* from the point of release, write expressions for the object's velocity $v_y(t)$ and position $y(t)$. **(b)** Find the time for the object to reach its highest point and its position y_{max} at that point. **(c)** Show that as the drag coefficient approaches zero, your last answer reduces to the well-known result $y_{max} = \frac{1}{2}v_o^2/g$ for an object in the vacuum. [*Hint:* If the drag force is very small, the terminal speed is very big, so v_o/v_{ter} is very small. Use the Taylor series for the log function to approximate $\ln(1+\delta)$ by $\delta - \frac{1}{2}\delta^2$. (For a little more on Taylor series see Problem 2.18.)]

2.12 ★★ Problem 2.7 is about a class of one-dimensional problems that can always be reduced to doing an integral. Here is another. Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,dx'. \tag{2.85}$$

[*Hint:* Use the chain rule to prove the following handy relation, which we could call the "$v\,dv/dx$ rule": If you regard v as a function of x, then

$$\dot{v} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}. \tag{2.86}$$

Use this to rewrite Newton's second law in the separated form $m\,d(v^2) = 2F(x)\,dx$ and then integrate from x_o to x.] Comment on your result for the case that $F(x)$ is actually a constant. (You may recognise your solution as a statement about kinetic energy and work, both of which we shall be discussing in Chapter 4.)

2.13 ★★ Consider a mass m constrained to move on the x axis and subject to a net force $F = -kx$ where k is a positive constant. The mass is released from rest at $x = x_o$ at time $t = 0$. Use the result (2.85) in Problem 2.12 to find the mass's speed as a function of x; that is, $dx/dt = g(x)$ for some function $g(x)$. Separate this as $dx/g(x) = dt$ and integrate from time 0 to t to find x as a function of t. (You may recognize this as one way — not the easiest — to solve the simple harmonic oscillator.)

2.14 ★★★ Use the method of Problem 2.7 to solve the following: A mass m is constrained to move along the x axis subject to a force $F(v) = -F_o e^{v/V}$, where F_o and V are constants. **(a)** Find $v(t)$ if the initial velocity is $v_o > 0$ at time $t = 0$. **(b)** At what time does it come instantaneously to rest? **(c)** By integrating $v(t)$, you can find $x(t)$. Do this and find how far the mass travels before coming instantaneously to rest.

SECTION 2.3 Trajectory and Range in a Linear Medium

2.15 ⋆ Consider a projectile launched with velocity (v_{xo}, v_{yo}) from horizontal ground (with x measured horizontally and y vertically up). Assuming no air resistance, find how long the projectile is in the air and show that the distance it travels before landing (the horizontal range) is $2v_{xo}v_{yo}/g$.

2.16 ⋆ A golfer hits his ball with speed v_o at an angle θ above the horizontal ground. Assuming that the angle θ is fixed and that air resistance can be neglected, what is the minimum speed v_o(min) for which the ball will clear a wall of height h, a distance d away? Your solution should get into trouble if the angle θ is such that $\tan\theta < h/d$. Explain. What is v_o(min) if $\theta = 25°$, $d = 50$ m, and $h = 2$ m?

2.17 ⋆ The two equations (2.36) give a projectile's position (x, y) as a function of t. Eliminate t to give y as a function of x. Verify Equation (2.37).

2.18 ⋆ Taylor's theorem states that, for any reasonable function $f(x)$, the value of f at a point $(x + \delta)$ can be expressed as an infinite series involving f and its derivatives at the point x:

$$f(x+\delta) = f(x) + f'(x)\delta + \frac{1}{2!}f''(x)\delta^2 + \frac{1}{3!}f'''(x)\delta^3 + \cdots \tag{2.87}$$

where the primes denote successive derivatives of $f(x)$. (Depending on the function this series may converge for *any* increment δ or only for values of δ less than some nonzero "radius of convergence.") This theorem is enormously useful, especially for small values of δ, when the first one or two terms of the series are often an excellent approximation.[11] **(a)** Find the Taylor series for $\ln(1 + \delta)$. **(b)** Do the same for $\cos\delta$. **(c)** Likewise $\sin\delta$. **(d)** And e^δ.

2.19 ⋆ Consider the projectile of Section 2.3. **(a)** Assuming there is no air resistance, write down the position (x, y) as a function of t, and eliminate t to give the trajectory y as a function of x. **(b)** The correct trajectory, including a linear drag force, is given by (2.37). Show that this reduces to your answer for part (a) when air resistance is switched off (τ and $v_{ter} = g\tau$ both approach infinity). [*Hint:* Remember the Taylor series (2.40) for $\ln(1 - \epsilon)$.]

2.20 ⋆⋆ [Computer] Use suitable graph-plotting software to plot graphs of the trajectory (2.36) of a projectile thrown at 45°above the horizontal and subject to linear air resistance for four different values of the drag coefficient, ranging from a significant amount of drag down to no drag at all. Put all four trajectories on the same plot. [*Hint:* In the absence of any given numbers, you may as well choose convenient values. For example, why not take $v_{xo} = v_{yo} = 1$ and $g = 1$. (This amounts to choosing your units of length and time so that these parameters have the value 1.) With these choices, the strength of the drag is given by the one parameter $v_{ter} = \tau$, and you might choose to plot the trajectories for $v_{ter} = 0.3$, 1, 3, and ∞ (that is, no drag at all), and for times from $t = 0$ to 3. For the case that $v_{ter} = \infty$, you'll probably want to write out the trajectory separately.]

2.21 ⋆⋆⋆ A gun can fire shells in any direction with the same speed v_o. Ignoring air resistance and using cylindrical polar coordinates with the gun at the origin and z measured vertically up, show that

[11] For more details on Taylor's series see, for example, Mary Boas, *Mathematical Methods in the Physical Sciences* (Wiley, 1983), p. 22 or Donald McQuarrie, *Mathematical Methods for Scientists and Engineers* (University Science Books, 2003), p. 94.

the gun can hit any object inside the surface

$$z = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2.$$

Describe this surface and comment on its dimensions.

2.22 ★★★ [Computer] The equation (2.39) for the range of a projectile in a linear medium cannot be solved analytically in terms of elementary functions. If you put in numbers for the several parameters, then it *can* be solved numerically using any of several software packages such as Mathematica, Maple, and MatLab. To practice this, do the following: Consider a projectile launched at angle θ above the horizontal ground with initial speed v_o in a linear medium. Choose units such that $v_o = 1$ and $g = 1$. Suppose also that the terminal speed $v_{ter} = 1$. (With $v_o = v_{ter}$, air resistance should be fairly important.) We know that in a vacuum, the maximum range occurs at $\theta = \pi/4 \approx 0.75$. **(a)** What is the maximum range in a vacuum? **(b)** Now solve (2.39) for the range in the given medium at the same angle $\theta = 0.75$. **(c)** Once you have your calculation working, repeat it for some selection of values of θ within which the maximum range probably lies. (You could try $\theta = 0.4, 0.5, \cdots, 0.8$.) **(d)** Based on these results, choose a smaller interval for θ where you're sure the maximum lies and repeat the process. Repeat it again if necessary until you know the maximum range and the corresponding angle to two significant figures. Compare with the vacuum values.

SECTION 2.4 Quadratic Air Resistance

2.23 ★ Find the terminal speeds in air of **(a)** a steel ball bearing of diameter 3 mm, **(b)** a 16-pound steel shot, and **(c)** a 200-pound parachutist in free fall in the fetal position. In all three cases, you can safely assume the drag force is purely quadratic. The density of steel is about 8 g/cm^3 and you can treat the parachutist as a sphere of density 1 g/cm^3.

2.24 ★ Consider a sphere (diameter D, density ϱ_{sph}) falling through air (density ϱ_{air}) and assume that the drag force is purely quadratic. **(a)** Use Equation (2.84) from Problem 2.4 (with $\kappa = 1/4$ for a sphere) to show that the terminal speed is

$$v_{ter} = \sqrt{\frac{8}{3}Dg\frac{\varrho_{sph}}{\varrho_{air}}}. \tag{2.88}$$

(b) Use this result to show that of two spheres of the same size, the denser one will eventually fall faster. **(c)** For two spheres of the same material, show that the larger will eventually fall faster.

2.25 ★ Consider the cyclist of Section 2.4, coasting to a halt under the influence of a quadratic drag force. Derive in detail the results (2.49) and (2.51) for her velocity and position, and verify that the constant $\tau = m/cv_o$ is indeed a time.

2.26 ★ A typical value for the coefficient of quadratic air resistance on a cyclist is around $c = 0.20$ N/(m/s)^2. Assuming that the total mass (cyclist plus cycle) is $m = 80$ kg and that at $t = 0$ the cyclist has an initial speed $v_o = 20$ m/s (about 45 mi/h) and starts to coast to a stop under the influence of air resistance, find the characteristic time $\tau = m/cv_o$. How long will it take him to slow to 15 m/s? What about 10 m/s? And 5 m/s? (Below about 5 m/s, it is certainly not reasonable to ignore friction, so there is no point pursuing this calculation to lower speeds.)

2.27 ★ I kick a puck of mass m up an incline (angle of slope = θ) with initial speed v_o. There is no friction between the puck and the incline, but there is air resistance with magnitude $f(v) = cv^2$. Write down and solve Newton's second law for the puck's velocity as a function of t on the upward journey. How long does the upward journey last?

2.28 ★ A mass m has speed v_o at the origin and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the "$v\,dv/dx$ rule" (2.86) in Problem 2.12 to write the equation of motion in the separated form $m\,v\,dv/F(v) = dx$, and then integrate both sides to give x in terms of v (or vice versa). Show that it will eventually travel a distance $2m\sqrt{v_o}/c$.

2.29 ★ The terminal speed of a 70-kg skydiver in spread-eagle position is around 50 m/s (about 115 mi/h). Find his speed at times $t = 1, 5, 10, 20, 30$ seconds after he jumps from a stationary balloon. Compare with the corresponding speeds if there were no air resistance.

2.30 ★ Suppose we wish to approximate the skydiver of Problem 2.29 as a sphere (not a very promising approximation, but nevertheless the kind of approximation physicists sometimes like to make). Given the mass and terminal speed, what should we use for the diameter of the sphere? Does your answer seem reasonable?

2.31 ★★ A basketball has mass $m = 600$ g and diameter $D = 24$ cm. **(a)** What is its terminal speed? **(b)** If it is dropped from a 30-m tower, how long does it take to hit the ground and how fast is it going when it does so? Compare with the corresponding numbers in a vacuum.

2.32 ★★ Consider the following statement: If at all times during a projectile's flight its speed is much less than the terminal speed, the effects of air resistance are usually very small. **(a)** Without reference to the explicit equations for the magnitude of v_{ter}, explain clearly why this is so. **(b)** By examining the explicit formulas (2.26) and (2.53) explain why the statement above is even more useful for the case of quadratic drag than for the linear case. [*Hint:* Express the ratio f/mg of the drag to the weight in terms of the ratio v/v_{ter}.]

2.33 ★★ The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as follows:

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

for any z, real or complex. **(a)** Sketch the behavior of both functions over a suitable range of real values of z. **(b)** Show that $\cosh z = \cos(iz)$. What is the corresponding relation for $\sinh z$? **(c)** What are the derivatives of $\cosh z$ and $\sinh z$? What about their integrals? **(d)** Show that $\cosh^2 z - \sinh^2 z = 1$. **(e)** Show that $\int dx/\sqrt{1+x^2} = \operatorname{arcsinh} x$. [*Hint:* One way to do this is to make the substitution $x = \sinh z$.]

2.34 ★★ The hyperbolic function $\tanh z$ is defined as $\tanh z = \sinh z/\cosh z$, with $\cosh z$ and $\sinh z$ defined as in Problem 2.33. **(a)** Prove that $\tanh z = -i\tan(iz)$. **(b)** What is the derivative of $\tanh z$? **(c)** Show that $\int dz \tanh z = \ln\cosh z$. **(d)** Prove that $1 - \tanh^2 z = \operatorname{sech}^2 z$, where $\operatorname{sech} z = 1/\cosh z$. **(e)** Show that $\int dx/(1-x^2) = \operatorname{arctanh} x$.

2.35 ★★ **(a)** Fill in the details of the arguments leading from the equation of motion (2.52) to Equations (2.57) and (2.58) for the velocity and position of a dropped object subject to quadratic air resistance. Be sure to do the two integrals involved. (The results of Problem 2.34 will help.) **(b)** Tidy the two equations by introducing the parameter $\tau = v_{\text{ter}}/g$. Show that when $t = \tau$, v has reached 76% of its terminal value. What are the corresponding percentages when $t = 2\tau$ and 3τ? **(c)** Show that when $t \gg \tau$, the position is approximately $y \approx v_{\text{ter}}t + \text{const}$. [*Hint:* The definition of $\cosh x$ (Problem 2.33) gives you a simple approximation when x is large.] **(d)** Show that for t small, Equation (2.58) for the position gives $y \approx \frac{1}{2}gt^2$. [Use the Taylor series for $\cosh x$ and for $\ln(1+\delta)$.]

2.36 ⋆⋆ Consider the following quote from Galileo's *Dialogues Concerning Two New Sciences:*

> Aristotle says that "an iron ball of 100 pounds falling from a height of one hundred cubits reaches the ground before a one-pound ball has fallen a single cubit." I say that they arrive at the same time. You find, on making the experiment, that the larger outstrips the smaller by two finger-breadths, that is, when the larger has reached the ground, the other is short of it by two finger-breadths.

We know that the statement attributed to Aristotle is totally wrong, but just how close is Galileo's claim that the difference is just "two finger breadths"? **(a)** Given that the density of iron is about 8 g/cm^3, find the terminal speeds of the two iron balls. **(b)** Given that a cubit is about 2 feet, use Equation (2.58) to find the time for the heavier ball to land and then the position of the lighter ball at that time. How far apart are they?

2.37 ⋆⋆ The result (2.57) for the velocity of a falling object was found by integrating Equation (2.55) and the quickest way to do this is to use the integral $\int du/(1-u^2) = \operatorname{arctanh} u$. Here is another way to do it: Integrate (2.55) using the method of "partial fractions," writing

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right),$$

which lets you do the integral in terms of natural logs. Solve the resulting equation to give v as a function of t and show that your answer agrees with (2.57).

2.38 ⋆⋆ A projectile that is subject to quadratic air resistance is thrown vertically *up* with initial speed v_o. **(a)** Write down the equation of motion for the upward motion and solve it to give v as a function of t. **(b)** Show that the time to reach the top of the trajectory is

$$t_{top} = (v_{ter}/g)\arctan(v_o/v_{ter}).$$

(c) For the baseball of Example 2.5 (with $v_{ter} = 35$ m/s), find t_{top} for the cases that $v_o = 1$, 10, 20, 30, and 40 m/s, and compare with the corresponding values in a vacuum.

2.39 ⋆⋆ When a cyclist coasts to a stop, he is actually subject to two forces, the quadratic force of air resistance, $f = -cv^2$ (with c as given in Problem 2.26), and a constant frictional force f_{fr} of about 3 N. The former is dominant at high and medium speeds, the latter at low speed. (The frictional force is a combination of ordinary friction in the bearings and rolling friction of the tires on the road.) **(a)** Write down the equation of motion while the cyclist is coasting to a stop. Solve it by separating variables to give t as a function of v. **(b)** Using the numbers of Problem 2.26 (and the value $f_{fr} = 3$ N given above) find how long it takes the cyclist to slow from his initial 20 m/s to 15 m/s. How long to slow to 10 and 5 m/s? How long to come to a full stop? If you did Problem 2.26, compare with the answers you got there ignoring friction entirely.

2.40 ⋆⋆ Consider an object that is coasting horizontally (positive x direction) subject to a drag force $f = -bv - cv^2$. Write down Newton's second law for this object and solve for v by separating variables. Sketch the behavior of v as a function of t. Explain the time dependence for t large. (Which force term is dominant when t is large?)

2.41 ⋆⋆ A baseball is thrown vertically up with speed v_o and is subject to a quadratic drag with magnitude $f(v) = cv^2$. Write down the equation of motion for the upward journey (measuring y vertically *up*) and show that it can be rewritten as $\dot{v} = -g[1 + (v/v_{ter})^2]$. Use the "$v\,dv/dx$ rule"

(2.86) to write $\dot{v}$ as $v\,dv/dy$, and then solve the equation of motion by separating variables (put all terms involving v on one side and all terms involving y on the other). Integrate both sides to give y in terms of v, and hence v as a function of y. Show that the baseball's maximum height is

$$y_{\text{max}} = \frac{v_{\text{ter}}^2}{2g} \ln\left(\frac{v_{\text{ter}}^2 + v_{\text{o}}^2}{v_{\text{ter}}^2}\right). \tag{2.89}$$

If $v_{\text{o}} = 20$ m/s (about 45 mph) and the baseball has the parameters given in Example 2.5 (page 61), what is y_{max}? Compare with the value in a vacuum.

2.42 ★★ Consider again the baseball of Problem 2.41 and write down the equation of motion for the downward journey. (Notice that with a quadratic drag the downward equation is different from the upward one, and has to be treated separately.) Find v as a function of y and, given that the downward journey starts at y_{max} as given in (2.89), show that the speed when the ball returns to the ground is $v_{\text{ter}}v_{\text{o}}/\sqrt{v_{\text{ter}}^2 + v_{\text{o}}^2}$. Discuss this result for the cases of very much and very little air resistance. What is the numerical value of this speed for the baseball of Problem 2.41? Compare with the value in a vacuum.

2.43 ★★★ [Computer] The basketball of Problem 2.31 is thrown from a height of 2 m with initial velocity $\mathbf{v}_{\text{o}} = 15$ m/s at 45° above the horizontal. **(a)** Use appropriate software to solve the equations of motion (2.61) for the ball's position (x, y) and plot the trajectory. Show the corresponding trajectory in the absence of air resistance. **(b)** Use your plot to find how far the ball travels in the horizontal direction before it hits the floor. Compare with the corresponding range in a vacuum.

2.44 ★★★ [Computer] To get an accurate trajectory for a projectile one must often take account of several complications. For example, if a projectile goes very high then we have to allow for the reduction in air resistance as atmospheric density decreases. To illustrate this, consider an iron cannonball (diameter 15 cm, density 7.8 g/cm^3) that is fired with initial velocity 300 m/s at 50 degrees above the horizontal. The drag force is approximately quadratic, but since the drag is proportional to the atmospheric density and the density falls off exponentially with height, the drag force is $f = c(y)v^2$ where $c(y) = \gamma D^2 \exp(-y/\lambda)$ with γ given by (2.6) and $\lambda \approx 10,000$ m. **(a)** Write down the equations of motion for the cannonball and use appropriate software to solve numerically for $x(t)$ and $y(t)$ for $0 \le t \le 3.5$ s. Plot the ball's trajectory and find its horizontal range. **(b)** Do the same calculation ignoring the variation of atmospheric density [that is, setting $c(y) = c(0)$], and yet again ignoring air resistance entirely. Plot all three trajectories for $0 \le t \le 3.5$ s on the same graph. You will find that in this case air resistance makes a huge difference and that the variation of air resistance makes a small, but not negligible, difference.

SECTION 2.6 Complex Exponentials

2.45 ★ **(a)** Using Euler's relation (2.76), prove that any complex number $z = x + iy$ can be written in the form $z = re^{i\theta}$, where r and θ are real. Describe the significance of r and θ with reference to the complex plane. **(b)** Write $z = 3 + 4i$ in the form $z = re^{i\theta}$. **(c)** Write $z = 2e^{-i\pi/3}$ in the form $x + iy$.

2.46 ★ For any complex number $z = x + iy$, the **real** and **imaginary parts** are defined as the real numbers $\text{Re}(z) = x$ and $\text{Im}(z) = y$. The **modulus** or **absolute value** is $|z| = \sqrt{x^2 + y^2}$ and the **phase** or **angle** is the value of θ when z is expressed as $z = re^{i\theta}$. The **complex conjugate** is $z^* = x - iy$. (This last is the notation used by most physicists; most mathematicians use $\bar{z}$.) For each of the following complex numbers, find the real and imaginary parts, the modulus and phase, and the complex conjugate, and sketch z and z^* in the complex plane:

$$\text{(a) } z = 1 + i \qquad \text{(b) } z = 1 - i\sqrt{3}$$
$$\text{(c) } z = \sqrt{2}e^{-i\pi/4} \qquad \text{(d) } z = 5e^{i\omega t}.$$

In part (d), ω is a constant and t is the time.

2.47 ⋆ For each of the following two pairs of numbers compute $z + w$, $z - w$, zw, and z/w.

$$\text{(a) } z = 6 + 8i \text{ and } w = 3 - 4i \qquad \text{(b) } z = 8e^{i\pi/3} \text{ and } w = 4e^{i\pi/6}.$$

Notice that for adding and subtracting complex numbers, the form $x + iy$ is more convenient, but for multiplying and especially dividing, the form $re^{i\theta}$ is more convenient. In part (a), a clever trick for finding z/w without converting to the form $re^{i\theta}$ is to multiply top and bottom by w^*; try this one both ways.

2.48 ⋆ Prove that $|z| = \sqrt{z^*z}$ for any complex number z.

2.49 ⋆ Consider the complex number $z = e^{i\theta} = \cos\theta + i\sin\theta$. **(a)** By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. **(b)** Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

2.50 ⋆ Use the series definition (2.72) of e^z to prove that[12] $de^z/dz = e^z$.

2.51 ⋆⋆ Use the series definition (2.72) of e^z to prove that $e^z e^w = e^{z+w}$. [*Hint:* If you write down the left side as a product of two series, you will have a huge sum of terms like $z^n w^m$. If you group together all the terms for which $n + m$ is the same (call it p) and use the binomial theorem, you will find you have the series for the right side.]

SECTION 2.7 Solution for the Charge in a B Field

2.52 ⋆ The transverse velocity of the particle in Sections 2.5 and 2.7 is contained in (2.77), since $\eta = v_x + iv_y$. By taking the real and imaginary parts, find expressions for v_x and v_y separately. Based on these expressions describe the time dependence of the transverse velocity.

2.53 ⋆ A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, **E** and **B**, both pointing in the z direction. The net force on the particle is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Write down the equation of motion for the particle and resolve it into its three components. Solve the equations and describe the particle's motion.

2.54 ⋆⋆ In Section 2.5 we solved the equations of motion (2.68) for the transverse velocity of a charge in a magnetic field by the trick of using the complex number $\eta = v_x + iv_y$. As you might imagine, the equations can certainly be solved without this trick. Here is one way: **(a)** Differentiate the first of equations (2.68) with respect to t and use the second to give you a second-order differential equation for v_x. This is an equation you should recognize [if not, look at Equation (1.55)] and you can write down its general solution. Once you know v_x, (2.68) tells you v_y. **(b)** Show that the general solution you get here is the same as the general solution contained in (2.77), as disentangled in Problem 2.52.

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[12] If you are the type who worries about mathematical niceties, you may be wondering if it is permissible to differentiate an infinite series. Fortunately, in the case of a power series (such as this), there is a theorem that guarantees the series can be differentiated for any z inside the "radius of convergence." Since the radius of convergence of the series for e^z is infinite, we can differentiate it for *any* value of z.

2.55 ★★★ A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields, **E** pointing in the y direction and **B** in the z direction (an arrangement called "crossed E and B fields"). Suppose the particle is initially at the origin and is given a kick at time $t = 0$ along the x axis with $v_x = v_{xo}$ (positive or negative). **(a)** Write down the equation of motion for the particle and resolve it into its three components. Show that the motion remains in the plane $z = 0$. **(b)** Prove that there is a unique value of v_{xo}, called the drift speed v_{dr}, for which the particle moves undeflected through the fields. (This is the basis of velocity selectors, which select particles traveling at one chosen speed from a beam with many different speeds.) **(c)** Solve the equations of motion to give the particle's velocity as a function of t, for arbitrary values of v_{xo}. [*Hint:* The equations for (v_x, v_y) should look very like Equations (2.68) except for an offset of v_x by a constant. If you make a change of variables of the form $u_x = v_x - v_{dr}$ and $u_y = v_y$, the equations for (u_x, u_y) will have exactly the form (2.68), whose general solution you know.] **(d)** Integrate the velocity to find the position as a function of t and sketch the trajectory for various values of v_{xo}.