

# Physics 411 - Winter 2015

Monday, week 2 :

## Newton's Laws

- We presented a glimpse into the history of the development of Newton's Laws
  - Aristotle → Descartes → Galileo → Newton
  - There are <sup>Courtiers</sup> numerous others who also contributed to our <sup>civilizations</sup> current understanding of the Laws of Motion. A paper exploring this history would make a fine course project.
    - Al Farabi
    - A rich history of science in Arab World (8<sup>th</sup>-13<sup>th</sup> century)
    - Indigenous "Americas"; Aztecs had most advanced Astronomy @ contact.
  - Newton had the Law of Inertia from Descartes & Galileo, but calculus got him the rest;

$$\vec{F}(t) = \vec{F}_0 + \vec{V}_0 dt + \frac{1}{2} \vec{a} dt^2 + \dots$$

$$\int dt \rightarrow dt$$

$$\vec{r}(t_0 + dt) = \vec{r}_0 + \vec{v}_0 dt + \frac{1}{2} \vec{a} dt^2$$

$$\vec{v}(t_0 + dt) = \vec{v}_0 + \vec{a} dt$$

So for fixed mass, Newton could actually define the force as proportional to  $\vec{a}$  & discover the nature of forces by studying the motion:

Forces  $\rightarrow$  Motion (space-time)

Body  
mass

mass is last property left to use  
in defining force:

$$m > 0$$

characterized resistance to change.

$$\begin{aligned} F &= m \\ \frac{d\vec{r}}{dt} &= \vec{v} \\ \frac{d^2\vec{r}}{dt^2} &= \vec{a} \\ \vec{a} &= \frac{d\vec{v}}{dt} \end{aligned}$$

(1)

So

$$\vec{F} = f(m) \vec{a}$$

Newton's Laws : (1687)

Descartes'

N.1 The Law of Inertia (Galileo's Principle of Inertia)

If  $\vec{F} = 0$ , then  $\vec{v} = \text{constant}$

"A Body at rest will remain at rest"

"A Body will continue in a ~~state of rest~~ state of rest

or with uniform, straight-line motion unless acted upon

by a force

since  $\vec{a} = 0$ ,  $\star$  implies this result.

$$\vec{a} = \frac{1}{m} (\vec{F} - m\vec{v})$$

N.2

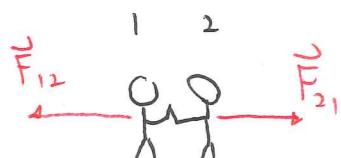
$$\begin{aligned}\vec{F} &\equiv \frac{d\vec{p}}{dt} \\ &= \frac{d}{dt}(m\vec{v}) \quad \xrightarrow{\text{linear momentum}} = m\vec{v} + m\vec{a} \\ &= m \frac{d\vec{v}}{dt} \quad \text{iff } m \neq m(t) \\ &= m\vec{a}\end{aligned}$$

Newton recognized that a changing mass w/ fixed force decreases/increases  $\vec{a}$ .

$$\vec{F} = \vec{F}(r, t)$$

of course,

$$N.3 \quad \sum m_i \vec{v}_i = \text{constant},$$



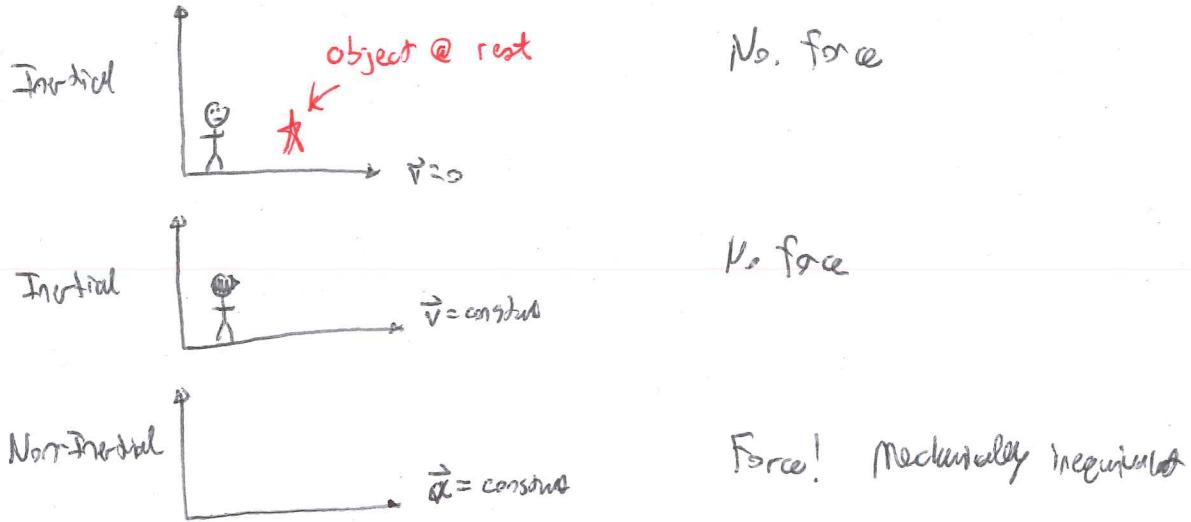
from applying  
N.2 to a system  
of particles with no net external force.

$$\vec{F}_{12} = -\vec{F}_{21}$$

"For every action, there is an equal  
& opposite reaction."

Remarks:

- $\vec{F} = m\vec{a}$  only holds for inertial reference frames



- In inertial reference frame, if  $\vec{F} = 0$  then N.2  $\Rightarrow$  N.1.

If all real forces (or sources of force) are known, then N.1 is used to determine inertial reference frames.

- $\vec{F} = m\vec{a} \Rightarrow$ 
    - $F_x(\vec{r}, t) = m \frac{d^2x}{dt^2}$
    - $F_y(\vec{r}, t) = m \ddot{y}$
    - $F_z(\vec{r}, t) = m \ddot{z}$
- $\# E_{\text{eq}} = \# \text{D.O.F}$   
for a point particle
- There are 3 Second-Order Differential Equations:
  - A particular solution then requires:
    - 2 initial conditions (per equation)
    - 2 non-trivial linearly independent ~~affin~~  
non-zero solutions (per eq.)

- In general, the # of D.E.Q = # D.O.F of the system. Any formalism must do this:  $\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{x}} = 0$  is  $\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \quad i=1, \dots, \text{D.O.F.}$

An Example to illustrate the general idea:

Constant force:  $F = ma$  (assume 1D)

$$= M \frac{d^2x}{dt^2}$$
$$= M \frac{dv}{dt}$$

Separation of variables

$$\int F/m dt = \int dv$$

$$F/m \cdot t + C_0 = V(t)$$

1<sup>st</sup> Initial Condition:

$$V_0 = C_0 + (F/m \cdot 0) \Rightarrow \boxed{V(t) = F/m \cdot t + V_0} \quad (1)$$
$$= dx/dt$$

Again...

$$\int dx = \int (F/m \cdot t + V_0) dt$$

$$x(t) = C_1 + \frac{1}{2} F/m t^2 + V_0 t$$

2<sup>nd</sup> Initial condition...

$$x(0) = x_0 = C_1 + 0 + 0$$

$$\Rightarrow \boxed{x(t) = x_0 + V_0 t + \frac{1}{2} F/m t^2} \quad (2)$$

Linearly Independent

Solve for  $t$  in (1) & replace in (2):  $v^2 - v_0^2 = 2 \left( \frac{F}{m} \right) (x - x_0)$

$$\text{or } \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2 = F \Delta x$$

In 3D,  $\frac{1}{2} m \vec{v}^2 - \frac{1}{2} m \vec{v}_0^2 = \vec{F} \cdot \vec{\Delta r}$

(4)

Even works for  $\vec{F} \neq \text{constant}$ , where  $\vec{F} = \text{constant}$  over  $\Delta\vec{r} \rightarrow d\vec{r}$  &  $dt \rightarrow dt$ :

$$\frac{1}{2}m\vec{v}^2 - \frac{1}{2}m\vec{v}_0^2 = \int_{t_0}^t \vec{F} \cdot d\vec{r}$$

Work-Kinetic Energy Theorem

- Conservation of Energy

$$T = \frac{1}{2}m\vec{v}^2 \quad \text{An important quantity}$$

We will see the homogeneity & isotropy of Space-Time, make

$$L = C\vec{v}^2$$

$C\vec{v}^2$  is apparently an important player in the motion of bodies.

No. 3  $\Rightarrow$  Conservation of linear momentum

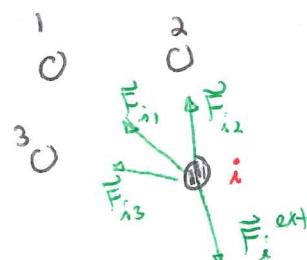
Consider a system of  $N$  particles ...

$$\vec{P}_{tot} = \sum_{i=1}^N \vec{p}_i$$

$$\dot{\vec{P}}_{tot} = \sum_{i=1}^N \dot{\vec{p}}_i$$

$$\dot{\vec{p}}_i = \sum_{j=1}^N \vec{F}_{ij} + \vec{F}_{i,ext}$$

$$\downarrow \vec{F}_{ii} = 0$$



$$\dot{\vec{P}}_{tot} = \sum_{i,j=1}^N \vec{F}_{ij} + \sum_{i=1}^N \vec{F}_{i,ext}$$

$$= \frac{1}{2} \sum_{i,j=1}^N (\vec{F}_{ij} + \vec{F}_{ji}) + \sum_{i=1}^N \vec{F}_{i,ext}$$

$$\downarrow \vec{F}_{ij} - \vec{F}_{ji} = 0 \text{ by N.3}$$

$$= \sum_{i=1}^N \vec{F}_{i,ext}$$

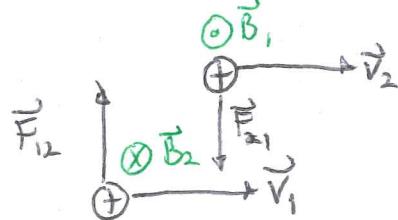
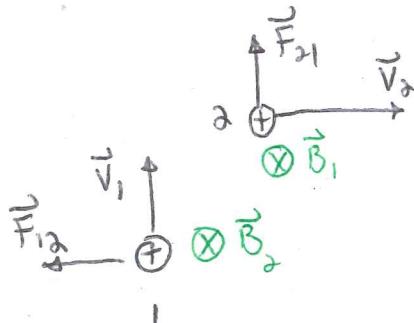
$$\dot{\vec{P}}_{tot} = \vec{F}_{tot,ext}$$

$$\text{So if } \vec{F}_{tot,ext} = 0 \Rightarrow \dot{\vec{P}}_{tot} = 0 \Rightarrow$$



$$\boxed{\vec{P}_{tot} = \sum_{i=1}^N \vec{p}_i = \text{constant}}$$

- Ex. of conservation of momentum where it appears to fail ?  $\vec{F} = q\vec{v} \times \vec{B}$



$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\vec{F}_{12} \neq -\vec{F}_{21}$$

$$\Rightarrow \sum \vec{p}_i \neq \text{constant}$$

↑  
Mechanical

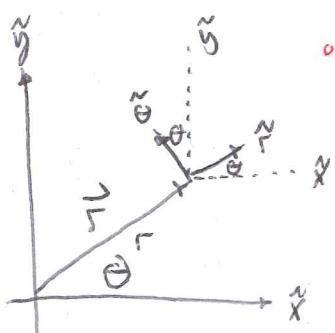
E.M field contains the missing momentum

But  $\sum \vec{L}_i \neq \text{constant}$

$\therefore \vec{L}_{\text{tot}}$  must be conserved in absence of torque, so

E.M field also has angular momentum

Let's look at No. 2 in polar coordinates, which will let us look @ S formalism:



$$x = r \cos \theta \quad \dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta$$

$$y = r \sin \theta \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta$$

$$\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}$$

$$\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}$$

$$\hat{r} \cdot \hat{\theta} = 0$$

$$|\hat{r}| = |\hat{\theta}| = 1$$

$$\ddot{r} = -\ddot{\theta} \sin \theta \hat{x} + \ddot{\theta} \cos \theta \hat{y} = \ddot{\theta} \hat{\theta}$$

$$\ddot{\theta} = -\ddot{\theta} \cos \theta \hat{x} - \ddot{\theta} \sin \theta \hat{y} = -\ddot{\theta} \hat{r}$$

We want  $\vec{F}$  for  $\vec{F} = m\vec{r}$

$$\vec{r} = r \hat{r}$$

$$\ddot{\vec{r}} = \ddot{r} \hat{r} + r \ddot{\hat{r}}$$

$$= \ddot{r} \hat{r} + r \ddot{\theta} \hat{\theta}$$

$$= \ddot{r} \hat{r} + r \omega \hat{\theta}$$

$$= V_r \hat{r} + V_\theta \hat{\theta}$$

$$\ddot{\vec{r}} = \ddot{r} \hat{r} + \ddot{\theta} \hat{\theta} + \ddot{\theta} \hat{r} + r \ddot{\theta} \hat{\theta} + r \ddot{\theta} \hat{\theta}$$

$$= \ddot{r} \hat{r} + \ddot{\theta} \hat{\theta} + \ddot{\theta} \hat{r} + r \ddot{\theta} \hat{\theta} - r \ddot{\theta}^2 \hat{r}$$

$$= (\ddot{r} - r \ddot{\theta}^2) \hat{r} + (2r \ddot{\theta} + r \ddot{\theta}) \hat{\theta}$$

radial centripetal

coriolis tangential

$$F_r = m \ddot{r} - m r \ddot{\theta}^2$$

$$F_\theta = 2m \ddot{r} \dot{\theta} + m r \ddot{\theta}$$

EOM in Polar

$$2m \vec{v} \times \vec{\omega} \sim q \vec{v} \times \vec{B}$$



(6)

# Physics 411 - Winter 2015

Wednesday, Week 2 3

## Review:

Newton's Laws

N.1 If  $\vec{F} = 0$ , then  $\vec{v} = \text{constant}$

N.2  $\vec{F} = d\vec{p}/dt$

N.3  $\vec{F}_{ij} = -\vec{F}_{Si} \Rightarrow \sum_{i=1}^N m_i \vec{v}_i = \sum_{i=1}^N \vec{p}_i$

- $\vec{F}(r, v, t) = m\vec{a}$  is a set of three PDE's for a <sup>single</sup> particle,
  - # D.O.F = # PDES
  - Solved  $\vec{F} = \text{constant}$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2$$

$$\vec{v} = \vec{v}_0 + \vec{a} t$$

$$\frac{1}{2} m \vec{v}^2 - \frac{1}{2} m \vec{v}_0^2 = \int_{t_0}^t \vec{F} \cdot d\vec{r}$$

- $\vec{F} = m\vec{a}$  in Polar Coordinates:

$$\vec{F} = m\ddot{\vec{r}} = \underbrace{(m\ddot{r} - mr\dot{\theta}^2)}_{\text{Centripetal}} \hat{r} + \underbrace{(2mr\dot{\theta})}_{\text{Coriolis}} \hat{\theta} - mr\ddot{\theta}\hat{r}$$

$2m\vec{v} \times \vec{\omega}$

## Eq. of Motion in Polar Coordinates using Lagrangian Approach :

$$\vec{F} = m\vec{a} \Leftrightarrow \frac{d\vec{L}}{dt} - \frac{d}{dt} \frac{\partial \vec{L}}{\partial \dot{q}} = 0 \quad \text{where } \vec{L} = T - V \\ = \text{K.E.} - \text{P.E.}$$

Consider a particle ..

$$x = r \cos \theta \quad \dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta$$

$$y = r \sin \theta \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta$$

$$T = \frac{1}{2} m \dot{r}^2$$

$$= \frac{1}{2} m [(\dot{r} \cos \theta - r \dot{\theta} \sin \theta)^2 + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta)^2]$$

$$= \frac{1}{2} m [\cancel{\dot{r}^2 \cos^2 \theta} + \cancel{r^2 \dot{\theta}^2 \sin^2 \theta} - 2 \cancel{r \dot{r} \dot{\theta} \cos \theta \sin \theta} \\ + \cancel{\dot{r}^2 \sin^2 \theta} + \cancel{r^2 \dot{\theta}^2 \cos^2 \theta} + 2 \cancel{r \dot{r} \dot{\theta} \sin \theta \cos \theta}]$$

$$= \frac{1}{2} m \dot{r}^2 + \underline{\frac{1}{2} m r^2 \dot{\theta}^2}$$

$$T \sim M$$

$$\omega \sim \sqrt{V}$$

$$\frac{1}{2} I \omega^2$$

Generalized Momentum

$$V = V(r, \theta)$$

$$q \rightarrow r \quad \frac{d\vec{L}}{dt} - \frac{d}{dt} \frac{\partial \vec{L}}{\partial \dot{r}} = m r \dot{\theta}^2 \cancel{- \frac{\partial U}{\partial r}} - \frac{d}{dt} (m \dot{r}) = 0$$

Generalized Force

$$m \ddot{r} - m r \dot{\theta}^2 = - \cancel{\frac{\partial U}{\partial r}} = - \vec{\nabla} U(\vec{r}) \cdot \vec{r}$$

Unit:

$$\text{Generalized momentum} = \vec{p} = \vec{r} \cdot \vec{\dot{r}}$$

Force

$$q \rightarrow \theta \quad \frac{d\vec{L}}{dt} - \frac{d}{dt} \frac{\partial \vec{L}}{\partial \dot{\theta}} = - \cancel{\frac{\partial U}{\partial \theta}} - \frac{d}{dt} (m r^2 \dot{\theta}) = - \cancel{\frac{\partial U}{\partial \theta}} - 2 m r \dot{r} \dot{\theta} - m r^2 \ddot{\theta} = 0$$

Torque

I &

$$\text{To forces} \Rightarrow 2 m \dot{r} \dot{\theta} + m r \ddot{\theta} = - \frac{1}{r} \frac{\partial U}{\partial \theta}$$

$$= - \vec{\nabla} U(\vec{r}) \cdot \hat{\theta}$$

Lagrangian method has gotten  
correct as well !!!

$$L = \frac{1}{2} r \ddot{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \ddot{\theta} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \dot{\theta}$$



So

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{Generalized Force}$$

$$\frac{\partial \mathcal{L}}{\partial q} = \text{Generalized Momentum}$$

Also, by equating the L approach with Newton's N.2 in polar:

$$-\vec{\nabla}U(\vec{r}) = \vec{F}$$

From the L, the quantity  $I = mr^2$  appears for a point particle  $\xi$ , from the Lagrangian eq. of motion we see:

$$I\ddot{\theta} = rF$$



$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$\alpha$  causes change in  $\theta$

we now have motion of  $\theta$  still driven by Force but we have  $m, r$  as free parameters  $\xi$ , both are appear as linear terms!

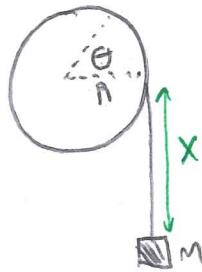
The full 3D relation  $\vec{I}\ddot{\theta} = \vec{r} \times \vec{F}$  will come out later when we see  $\vec{p}, \vec{r}, \vec{L}$  are naturally conserved quantities.

Let's state Newton's Laws for Rotations:

N.O.1 If  $\vec{r} = \vec{r} \times \vec{F} = 0 \Rightarrow \vec{\omega} = \text{constant}$

N.O.2  $\vec{r} = \vec{r} \times \vec{F} = \frac{d}{dt} \vec{L} = \frac{d}{dt} (\vec{r} \times \vec{p})$   
 $= \underline{\underline{\tau}} \vec{q}$

Ex)



Newton:

$$(1) M\ddot{x} = Mg - T$$

$$(2) I\ddot{\theta} = -R\ddot{T} \rightarrow \text{remember } \textcircled{1} \text{ is negative Torque}$$

$$\begin{aligned} -\theta R &= x \\ -\ddot{\theta} R &= \ddot{x} \end{aligned}$$

$$-I_{\beta^2} \ddot{x} = -R\ddot{T}$$

↓ w/ (1)

$$M\ddot{x} = Mg - I_{\beta^2} \ddot{x} \Rightarrow \ddot{x}(M + I_{\beta^2}) = Mg$$

$$\boxed{\ddot{x} = \frac{M}{M + I_{\beta^2}} \cdot g}$$

Lagrangian:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} M \dot{x}^2 + Mgx \\ &= \frac{1}{2} \frac{I}{R^2} \dot{x}^2 + \frac{1}{2} M \dot{x}^2 + Mgx \end{aligned}$$

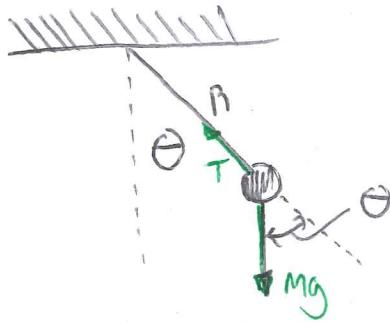
$$= \frac{1}{2} \left( \frac{I}{R^2} + M \right) \dot{x}^2 + Mgx$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = Mg - \frac{d}{dt} \left( \underbrace{\left( \frac{I}{R^2} + M \right) \dot{x}}_{\text{Generalized momentum}} \right)$$

Generalized momentum

$$= Mg - \left( \frac{I}{R^2} + M \right) \ddot{x} \Rightarrow \boxed{\ddot{x} = \frac{M}{M + I/R^2} g}$$

Ex)



Newton in Polar:

$$F_r = m\ddot{r} - m\dot{r}\dot{\theta}^2 \Rightarrow -m\dot{r}\dot{\theta}^2 = mg\cos\theta - T \\ = mg\cos\theta - T$$

$$F_\theta = 2m\dot{r}\dot{\theta} + m\dot{r}\ddot{\theta} \Rightarrow m\dot{r}\ddot{\theta} = -mg\sin\theta \\ = -mg\sin\theta$$

$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

Lagrangian:

$$L = T - V$$

$$= \frac{1}{2}MR^2\dot{\theta}^2 - mgR(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = -mgR\sin\theta - \frac{d}{dt}(MR^2\dot{\theta})$$

Generalized momenta  
Iw

Generalized force:

Torque  $\vec{r} \times \vec{F}$ !

$$= -mgR\sin\theta - MR^2\ddot{\theta}$$

$$= 0$$

$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

$$\ddot{\theta} \approx -\frac{g}{m}\theta$$

We've seen the relationship between  $\vec{r}$  &  $\phi$

Is  $\phi$  a vector?

$$\vec{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

$m$

$$\vec{p} = m\vec{v}$$

$$\phi$$

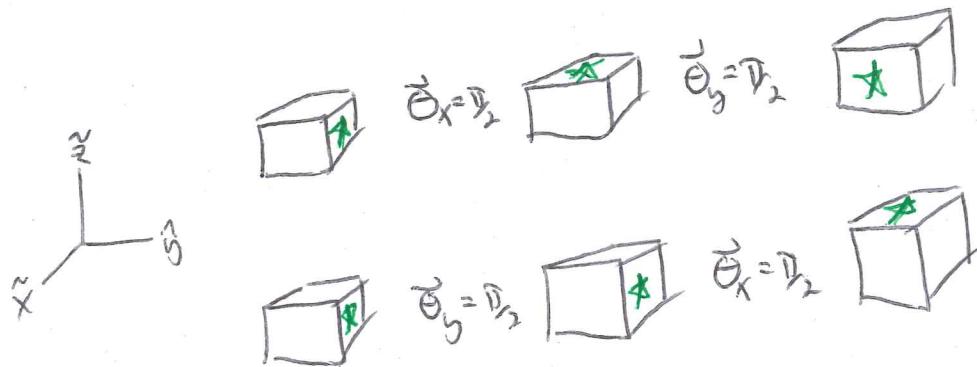
$$\vec{w}$$

$$\frac{d\vec{w}}{dt}$$

$$I = Mr^2 ; \hat{I}$$

$$\vec{L} = \hat{I}\vec{\omega} ; \hat{\vec{L}} = I\vec{w}$$

Is  $\phi$  a vector like  $\vec{r}$ ?



$$\text{So } \vec{\theta}_x + \vec{\theta}_y \neq \vec{\theta}_y + \vec{\theta}_x \Rightarrow \vec{\theta} \text{ not a vector!!}$$

Recall  $T_\theta$  rotation matrices

$$T_{\theta_x} T_{\theta_y} \neq T_{\theta_y} T_{\theta_x}$$

Why then is  $\vec{w} = \frac{d\vec{\theta}}{dt}$  a vector?

Consider a vector  $\vec{r}$  (any vector) rotating about an arbitrary direction  $\hat{e}_\theta$

# Physics 411 - winter 2015

Friday, Week 2:

Review:

Euler-Lagrange Equation for EOM:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 \quad q \rightarrow r, \theta$$

$$\vec{r}(r, \theta)$$

$$\frac{\partial \mathcal{L}}{\partial q} = \text{Generalized Force}$$



$$r \quad \theta$$

Force [N]

Torque [Nm]

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{Generalized Momentum}$$

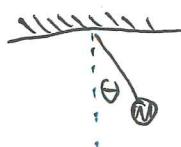
Liner Momentum

Angular momentum

$$-\vec{\nabla}U(\vec{r}) = \vec{F}$$

N.O.1 If  $\vec{P} = \vec{r} \times \vec{F} = 0$ , then  $\vec{\omega} = \text{constant}$

N.O.2  $\vec{P} = \vec{r} \times \vec{F}$   
 $= \frac{d\vec{r}}{dt} = \frac{d}{dt} \vec{r}$



$$\vec{F} = m\vec{a} \Rightarrow \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(m\vec{\omega})$$

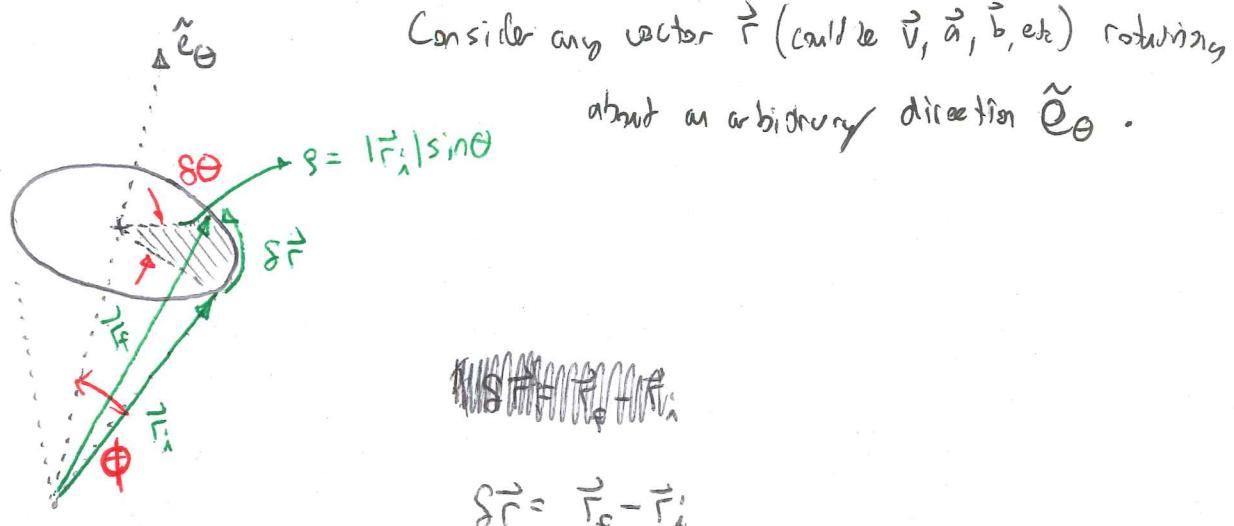
Force  $\rightarrow \frac{\partial \mathcal{L}}{\partial q}$  Motion  $(\vec{r}, \vec{\omega})$

Momentum  $\rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}}$

Energy  $\rightarrow \frac{\partial \mathcal{L}}{\partial t}$   $\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$

$\rightarrow T_0$  discrete rotations.

- $\vec{\omega}$  is not a vector:  $\vec{\theta}_x + \vec{\theta}_y \neq \vec{\theta}_y + \vec{\theta}_x$  {addition of rotation is not commutative}
- Why is  $\vec{\omega} = d\vec{\phi}/dt$  a vector?



$$\begin{aligned}
 |\delta \vec{r}| &= |\vec{r}_f - \vec{r}_i| \\
 &= |\tilde{e}_\theta \times \vec{r}_i| / \sin \theta \\
 &= s \theta \times \vec{r}_i
 \end{aligned}$$

Then  $\vec{r}_f = \vec{r}_i + \delta \vec{r}$

$$\begin{aligned}
 &= \vec{r}_i + s \theta \times \vec{r}_i
 \end{aligned}$$

Consider two rotations about two different directions  $\tilde{e}_1, \tilde{e}_2$ :

$$1 \rightarrow 2: \quad \vec{r}_f = \vec{r} + s \theta_1 \times \vec{r}$$

$$\begin{aligned}
 \vec{r}_f &= \vec{r}_{12} \quad \text{[cancel]} \\
 &= \vec{r} + s \theta_1 \times \vec{r} + s \theta_2 \times (\vec{r} + s \theta_1 \times \vec{r}) \\
 &= \vec{r} + s \theta_1 \times \vec{r} + s \theta_2 \times \vec{r} + s \theta_2 \times (s \theta_1 \times \vec{r}) \\
 &\quad \text{[cancel]}
 \end{aligned}$$

$$2 \rightarrow 1: \quad \vec{r}_f = \vec{r} + s \theta_1 \times \vec{r} + s \theta_2 \times \vec{r} + s \theta_1 \times (s \theta_2 \times \vec{r})$$

$$\begin{aligned}
 &= \vec{r}_{21}
 \end{aligned}$$

but as  $s \theta_1, s \theta_2 \rightarrow d\theta_1, \vec{r}_{12} = \vec{r}_{21}$

so  $s \theta$  is a vector for infinitesimal rotations.

(2)

Then,

$$\vec{S}_r = \vec{S}\theta \times \vec{r}$$

Again, any vector  $\vec{r}$

so  $\frac{\vec{S}_r}{dt} = \frac{\vec{S}\theta}{dt} \times \vec{r}$  or  $\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$

Because  $\vec{S}_r = (\vec{S}\theta_1 + \vec{S}\theta_2) \times \vec{r}$

$$= (\vec{S}\theta_2 + \vec{S}\theta_1) \times \vec{r}$$

then  $\vec{v} = (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{r}$

$$= (\vec{\omega}_2 + \vec{\omega}_1) \times \vec{r} \Rightarrow \vec{\omega} \text{ is a vector.}$$

$\vec{q} = \vec{\omega}$  is a vector.

Now, how will we solve ODEs like...  $\ddot{\theta} = -\frac{g}{l} \theta$ :

- By Separation of Variables; by integration

- Power Series:  $\theta(t) = \sum_{n=0}^{\infty} a_n t^n$

- Special Functions:  $x^n, \sin x, \cos x, e^{nx}$

- Fourier Transform or Laplace Transform:

$$\frac{d^2 f}{dt^2} + C^2 f = g \quad f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$\int [-\omega^2 f(\omega) + C^2 F(\omega)] e^{i\omega t} d\omega = \int f(t) e^{i\omega t} dt$$

$$F(\omega) = \frac{f(\omega)}{C^2 - \omega^2}$$

- Computationally

- Inspiration, Guessing! Find 2 linearly independent soln & we're done.

(3)

$$\text{Ex] } \ddot{\theta}(t) = -\frac{\beta}{\beta} \theta(t) \quad \theta(t) = C e^{\gamma t}$$

$\text{Thm, } \gamma^2 C e^{\gamma t} = -\frac{\beta}{\beta} C e^{\gamma t} \Rightarrow \gamma^2 = -\frac{\beta}{\beta} = -\omega^2$

$$\gamma = \pm i \sqrt{\frac{\beta}{\beta}} = \pm i \omega$$

$$\theta_1(t) = C_1 e^{i\omega t}$$

$$\theta_2(t) = C_2 e^{-i\omega t} \quad \text{but} \quad \ddot{\theta}^*(t) = -\frac{\beta}{\beta} \theta^* \Rightarrow \theta_1 = \theta_2^*$$

Linearly Independent

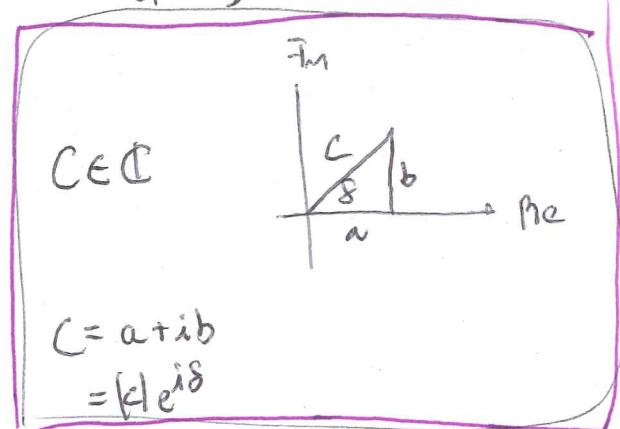
$$\begin{aligned} \Rightarrow \theta(t) &= C_1 e^{i\omega t} + C_2^* e^{-i\omega t} \\ &= 2 \operatorname{Re} [C_1 e^{i\omega t}] \\ &= 2 \operatorname{Re} [|C| e^{i(\omega t + \phi)}] \\ &= 2 |C| \cos(\omega t + \phi) \end{aligned}$$

$$\boxed{\theta(t) = A \cos(\omega t + \phi)}$$

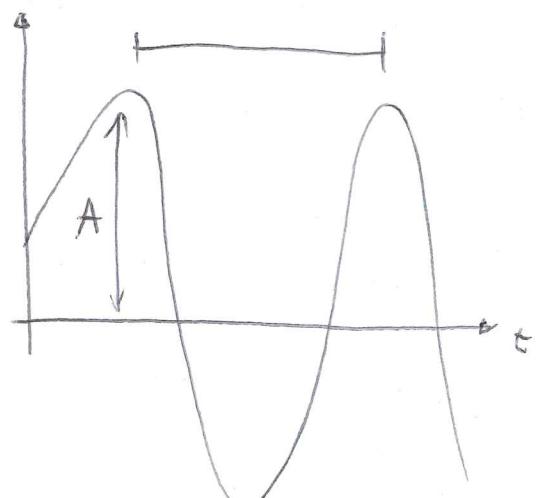
$$= A \cos \omega t \cos \phi - A \sin \omega t \sin \phi$$

$$= C \cos \omega t + D \sin \omega t$$

Linearly Independent



$$\theta(t) \quad T = \frac{2\pi}{\omega} = \frac{1}{f}$$

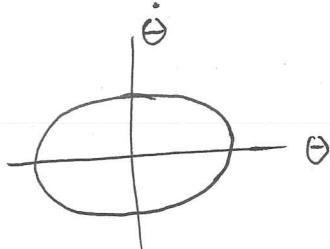


Mathematica to solve  $\ddot{\theta} = -\frac{g}{l} \sin \theta$  computationally:

- Plot function
- NDSolve

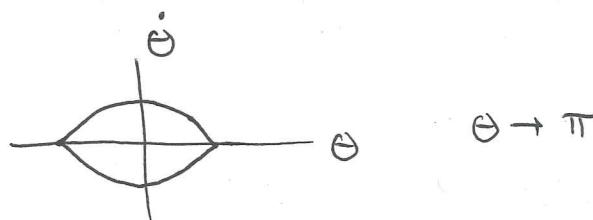
- $\theta = \theta_0 \sin wt$
- $\dot{\theta} = \theta_0 w \cos wt$

$$\Rightarrow (\dot{\theta}) = \begin{pmatrix} \theta \\ \theta_0 w \cos wt \end{pmatrix}$$



Elliptical when  $\theta \sim \sin wt$

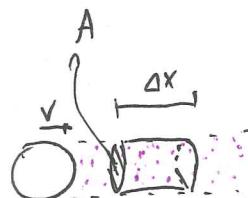
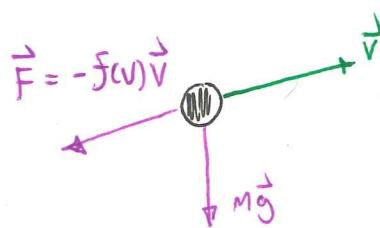
$$\theta \ll 1$$



$$\theta \rightarrow \pi$$

More  $\vec{F} = m\vec{a}$

- Motion of bodies in a gravitational field + air resistance



$$f(v) = a + bv + cv^2 + \dots$$

$$3\pi\eta DV$$

Viscosity

Diameter  $D$

Inertial Drag

$$\Delta M = A \cdot \Delta x \rho$$

$$\frac{dM}{dt} = A \rho v$$

$$\begin{aligned} F &= \frac{d}{dt}(Mv) = \frac{dM}{dt}v + M \frac{dv}{dt} \\ &= A \rho v^2 \\ &= \frac{\pi}{4} \rho D^2 v^2 \end{aligned}$$

Stokes' Drag

(5)