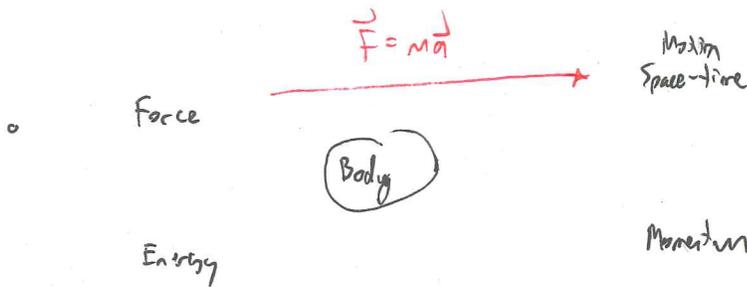
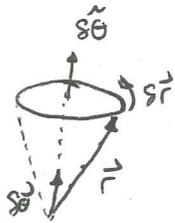


Physics 411 - Winter 2015

Wednesday, Week 3:

• Review:

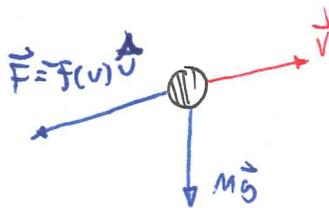
$$\delta \vec{r} = \delta \vec{\theta} \times \vec{r} \quad \Rightarrow \quad \dot{\vec{v}} = \dot{\vec{\omega}} \times \vec{r}$$



$$\theta(t) = \theta_0 \cos(\omega t + \phi) \quad \text{by guessing } e^{\lambda t}$$

• NDSolve in Mathematica

• More Example:



$$f(v) = a + bv + cv^2 + \dots$$

$$3 \pi \eta D v$$

$$K \frac{\pi}{4} S D^2 v^2$$

Stokes Drag

Inertial Drag

In air for spheres ...

$$f_{\text{lin}} = bV$$

$$= \beta DV$$

$$\downarrow$$

$$1.6 \times 10^{-4} \text{ N}\cdot\text{s}/\text{m}^2$$

$$f_{\text{quad}} = cV^2$$

$$= \gamma D^2 V^2$$

$$\downarrow$$

$$0.25 \text{ N}\cdot\text{s}^2/\text{m}^2$$

which one dominates?

Reynold's Number

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx \frac{\gamma D^2 V^2}{\eta DV} = \frac{\gamma DV}{\eta} = \frac{\gamma}{\beta} DV$$

$$= 1.6 \times 10^3 DV$$

∴ $f_{\text{quad}} \gg f_{\text{lin}}$ for most objects where $D \cdot V > 1.6 \times 10^{-3}$

Ex] $D = D_{\text{raindrop}} = 1 \text{ mm}$ $V = 1 \text{ m/s}$ $f_{\text{quad}} \neq f_{\text{lin}}$

Linear Air Resistance: ($\vec{F} = -bV\hat{v}$)

$$\vec{F} = m\vec{a} \Rightarrow m\ddot{\vec{r}} = -bV\hat{v} - mg\hat{y}$$

$$= -b\vec{v} - mg\hat{y}$$

$$m\dot{\vec{v}} = -b\vec{v} - mg\hat{y}$$

$$m \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} -bv_x \\ -bv_y - mg \end{pmatrix}$$

\hat{x}_0

$$\dot{v}_x = -\frac{b}{m} v_x$$

$$\dot{v}_y = -\frac{b}{m} v_y$$

$$\frac{dv}{dt} = -\frac{b}{m} v$$

$$\int \frac{dv}{v} = -\int \frac{b}{m} dt$$

$$\ln v = -\frac{b}{m} t + C$$

C

$$v(t) = \exp(-\frac{b}{m}t + C)$$

$$= D e^{-b/m t}$$

$$v(0) = v_0$$

$$= D$$

$$v_x(t) = v_0 e^{-b/m t}$$

$$\frac{dx}{dt} = v_0 e^{-t/\tau}$$

Characteristic Time

$$\tau \equiv \frac{m}{b}$$

$$\tau \rightarrow \frac{1}{3} v_0$$

$$3\tau \rightarrow \frac{1}{27} v_0 \approx 4\% v_0$$

Final V

$$\int dx = \int v_0 e^{-t/\tau} dt$$

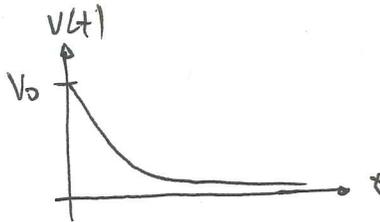
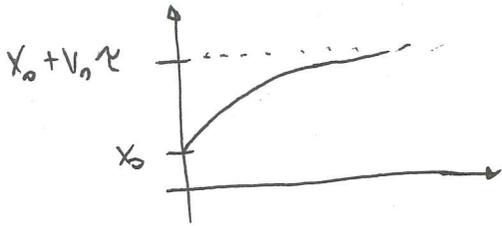
$$x = -v_0 \tau e^{-t/\tau} + C$$

$$x(0) = x_0 = -v_0 \tau + C$$

$$x(t) = x_0 + v_0 \tau (1 - e^{-t/\tau})$$

2

6



\tilde{y}_0

$$m\dot{v} = -(mg + bv)$$

$$= -(-bV_{\text{term}} + bv)$$

When $\dot{v} = 0$

$$V_{\text{term}} = \frac{-mg}{b}$$

$$= -g \cdot \tau$$

$$\dot{v} = -\frac{b}{m}(v - V_{\text{term}})$$

$$\int \frac{dv}{v - V_{\text{term}}} = -\int \frac{b}{m} dt$$

Note: $\tau \equiv \frac{m}{b} \propto \frac{D^3}{D} = D^2$

$$V_{\text{term}} = g\tau \propto D^2$$

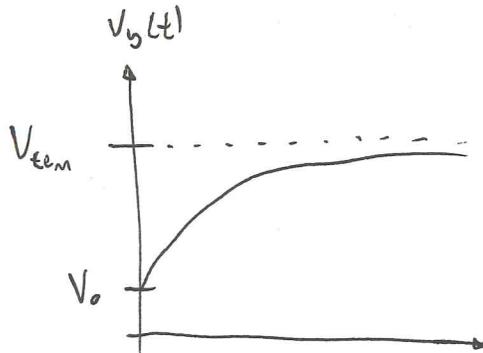
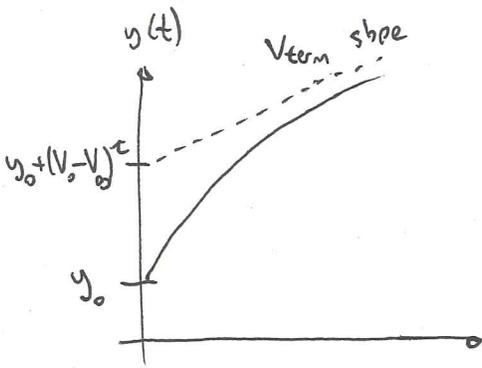
$$v(t) = (v_0 - V_{\text{term}})e^{-t/\tau} + V_{\text{term}}$$

Small spheres (rain drops) reach $V_{\text{term}} \rightarrow 0$ very fast ($\tau \rightarrow 0$) \Rightarrow MIST.

$$= dy/dt$$

$$\int dy = \int ((v_0 - V_{\text{term}})e^{-t/\tau} + V_{\text{term}}) dt$$

$$y(t) = y_0 + V_{\text{term}} \cdot t + (v_0 - V_{\text{term}})\tau(1 - e^{-t/\tau})$$



Quadratic Air Resistance :

$$m \ddot{\vec{r}} = -mg \hat{y} - c v^2 \hat{v}$$

$$m \dot{\vec{v}} = -mg \hat{y} - c v \vec{v}$$

$$\begin{pmatrix} m \dot{v}_x \\ m \dot{v}_y \end{pmatrix} = \begin{pmatrix} -c (v_x^2 + v_y^2)^{1/2} v_x \\ -mg - c (v_x^2 + v_y^2)^{1/2} v_y \end{pmatrix} \rightarrow \text{Hard!!}$$

Consider simple cases :

$$m \dot{v}_x = -c v_x^2 \quad \rightarrow |v_x| v_x$$

$$m \dot{v}_y^2 = -mg - c |v_y| v_y$$

\hat{x} :

$$m \frac{dv}{dt} = -c v^2 \rightarrow \int \frac{dv}{v^2} = -\int \frac{c}{m} dt \rightarrow \left. \frac{-1}{v} \right|_{v_0}^{v_f} = -\frac{c}{m} t$$

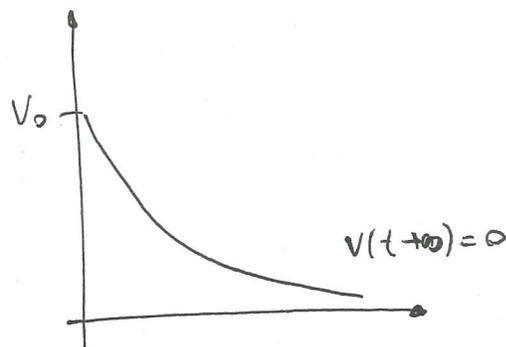
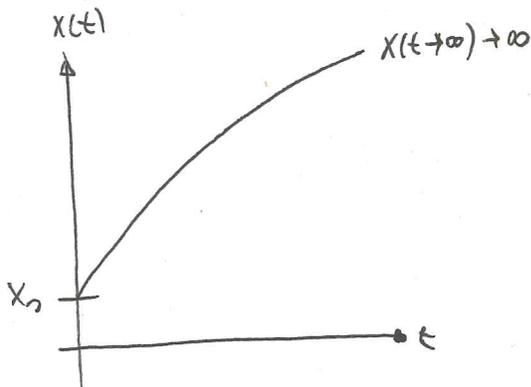
○

$$v_x(t) = \frac{v_0}{1 + t/\tau}$$

$$\tau \equiv \frac{m}{c v_0}$$

$$\int dx = \int \frac{v_0}{1 + t/\tau} dt$$

$$x(t) = x_0 + v_0 \tau \ln(1 + t/\tau)$$



Not physical, since $v \ll 1 \Rightarrow F_{\text{linear}}$ dominates.

Optional

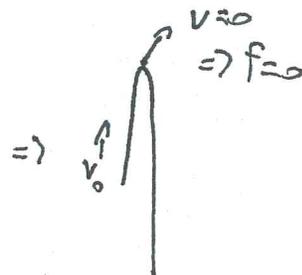
y:

$$m \dot{v}_y = -mg - c v_y^2$$

~~for $v_y \neq 0$~~

$$\begin{aligned} \dot{v}_y &= -g - \frac{c}{m} v_y^2 \\ &= -g \left(1 + \frac{c}{mg} v^2 \right) \end{aligned}$$

$$\dot{v}_y = 0 \Rightarrow \frac{v^2}{\tan} = \frac{-mg}{c} \Rightarrow v = \frac{1}{\tan} \sqrt{\frac{mg}{c}}$$



Prop. streng
Spannrest
 \Rightarrow



$$m \ddot{y} = -mg + c v^2$$

$$\begin{aligned} \dot{v} &= -g + \frac{c}{m} v^2 \\ &= -g \left(1 - \frac{c}{mg} v^2 \right) \\ &= -g \left(1 - \frac{v^2}{v_{\tan}^2} \right) \end{aligned}$$

$$v_{\tan} = \sqrt{\frac{mg}{c}}$$

$$\frac{dv}{1 - \frac{v^2}{v_{\tan}^2}} = -g dt$$

$$\frac{v_{\tan} dy}{1 - u^2} = -g dt \quad u = \frac{v}{v_{\tan}}$$

$$\begin{aligned} v_{\tan} \frac{1}{2} \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) dy &= -g t \\ \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du &= -\frac{2gt}{v_{\tan}} \end{aligned}$$

Optional

$$\ln(1+u) - \ln(1-u) = -\frac{2gt}{V_0} + 2C$$

$$\ln\left(\frac{1+u}{1-u}\right)$$

$$= 2f(t)$$

$$f(t) = \frac{-gt}{V_0} + C$$

$$= -\frac{t}{\tau} + C$$

$$\frac{1+u}{1-u} = e^{2f(t)}$$

$$1+u = (1-u)e^{2f}$$

$$u + ue^{2f} = e^{2f} - 1$$

$$u(1+e^{2f}) = e^{2f} - 1$$

$$u = \frac{e^{2f} - 1}{e^{2f} + 1} \cdot \begin{pmatrix} e^{-f} \\ e^{-f} \end{pmatrix}$$

$$= \frac{e^f - e^{-f}}{e^f + e^{-f}}$$

$$= D e^{\frac{t}{\tau}}$$

$$= D \frac{e^{\frac{t}{\tau}} - e^{-\frac{t}{\tau}}}{e^{\frac{t}{\tau}} + e^{-\frac{t}{\tau}}}$$

$$\frac{1}{2}(e^x - e^{-x}) = \sinh x$$

$$\frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\frac{1}{2}(e^{ix} - e^{-ix}) = \sin x$$

$$\frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$$

$$v(t) = V_0 \cdot D \tanh\left(\frac{t}{\tau}\right)$$

$$t \rightarrow \infty \Rightarrow D=1$$

$$v(t) = V_0 \tanh\left(\frac{t}{\tau}\right)$$

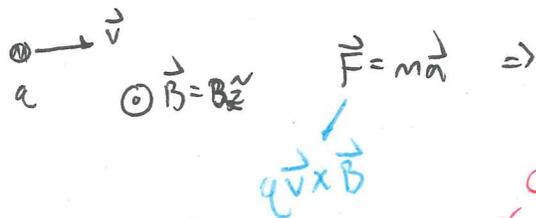
$$y(t) = \int V_0 \frac{\sinh(\frac{t}{\tau})}{\cosh(\frac{t}{\tau})} dt$$

$$x = \cosh \frac{t}{\tau} \quad dx = \frac{1}{\tau} \sinh \frac{t}{\tau} dt$$

$$= V_0 \cdot \tau \ln\left(\cosh \frac{t}{\tau}\right)$$

$$= \frac{V_0^2}{g} \ln\left(\cosh \frac{t}{\tau}\right)$$

Ex 1



$$m\ddot{\vec{v}} = q \begin{pmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & 0 & B \end{pmatrix} = q \begin{pmatrix} v_y B \\ -v_x B \\ 0 \end{pmatrix} = m \begin{pmatrix} dv_x/dt \\ dv_y/dt \\ 0 \end{pmatrix}$$

$$\ddot{\vec{v}} = \frac{qB}{m} \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix}$$

$$= \omega \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix}$$

$$\Rightarrow \ddot{\vec{v}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \vec{v}$$

$$\boxed{\ddot{\vec{v}} = M\vec{v}} \quad (1)$$

Solution:

Guess $\vec{v} = \vec{A} e^{i\lambda t}$ $\vec{A} \in \mathbb{C}^2$

$$(1) \Rightarrow \lambda \vec{A} e^{i\lambda t} = M \vec{A} e^{i\lambda t} \quad \text{or} \quad (M - I\lambda) \vec{A} = \vec{0}$$

want $\vec{A} \neq 0 \Rightarrow \det(M - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{vmatrix} = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$$

Now for the eigenvectors:

$$+i\omega \begin{pmatrix} -i\omega & \omega \\ -\omega & -i\omega \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = 0 \Rightarrow A_y = i A_x \quad \text{or} \quad \vec{A}_+ = \begin{pmatrix} 1 \\ i \end{pmatrix} A_+ e^{i\omega t}$$

$$-i\omega \begin{pmatrix} i\omega & \omega \\ -\omega & i\omega \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = 0 \Rightarrow A_y = -i A_x \quad \text{or} \quad \vec{A}_- = \begin{pmatrix} 1 \\ -i \end{pmatrix} A_- e^{i\omega t}$$

Solutions:

+i ω :

$$\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} A e^{i(\omega t + \phi)}$$

$$= A e^{i(\omega t + \phi)} \tilde{x} + i A e^{i(\omega t + \phi)} \tilde{y}$$

$$i = e^{i\pi/2}$$
$$= A e^{i(\omega t + \phi)} \tilde{x} + A e^{i(\omega t + \phi + \pi/2)} \tilde{y}$$

Take Re Part

$$\vec{v} = A \begin{pmatrix} \cos(\omega t + \phi) \\ \cos(\omega t + \phi + \pi/2) \end{pmatrix}$$

$$\vec{v} = A \begin{pmatrix} \cos(\omega t + \phi) \\ -\sin(\omega t + \phi) \end{pmatrix}$$

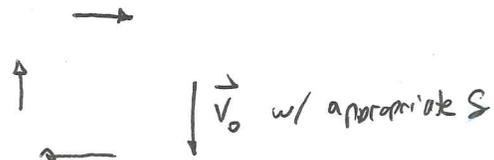
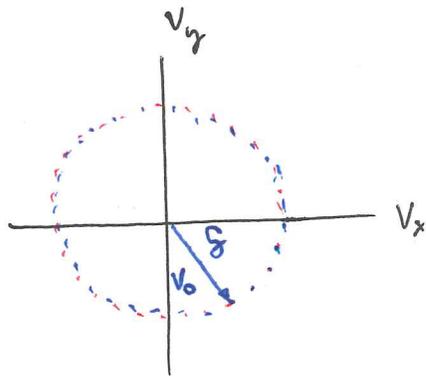
-i ω sol'n is similar.

Initial Condition:

$$\vec{v}(0) = \vec{v}_0 = A \begin{pmatrix} \cos \phi \\ -\sin \phi \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$|v_0| = |A| \quad v_x, v_y \text{ contained in } \phi.$$

$$\vec{v}(t) = v_0 \begin{pmatrix} \cos(\omega t + \phi) \\ -\sin(\omega t + \phi) \end{pmatrix}$$



Then $\vec{x}(t) \dots$

$$\begin{aligned}\vec{x}(t) &= \int \vec{v}(t) dt \\ &= \frac{V_0}{\omega} \begin{pmatrix} \sin(\omega t + \delta) \\ \cos(\omega t + \delta) \end{pmatrix} + \vec{C}\end{aligned}$$

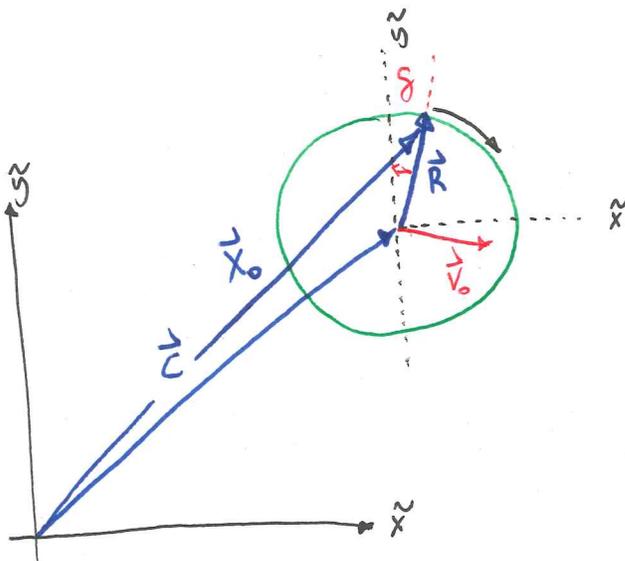
Integration constant

$$\begin{aligned}\vec{x}_0 &= \frac{V_0}{\omega} \begin{pmatrix} \sin \delta \\ \cos \delta \end{pmatrix} + \vec{C} \\ &= \cancel{\text{Hatched}} \vec{R} + \vec{C}\end{aligned}$$

$$R = \frac{V_0}{\omega} = \frac{mV_0}{qB}$$

$$\vec{C} = \vec{x}_0 - \vec{R}$$

$$\vec{x}(t) = R \begin{pmatrix} \sin(\omega t + \delta) \\ \cos(\omega t + \delta) \end{pmatrix} + \vec{C}$$

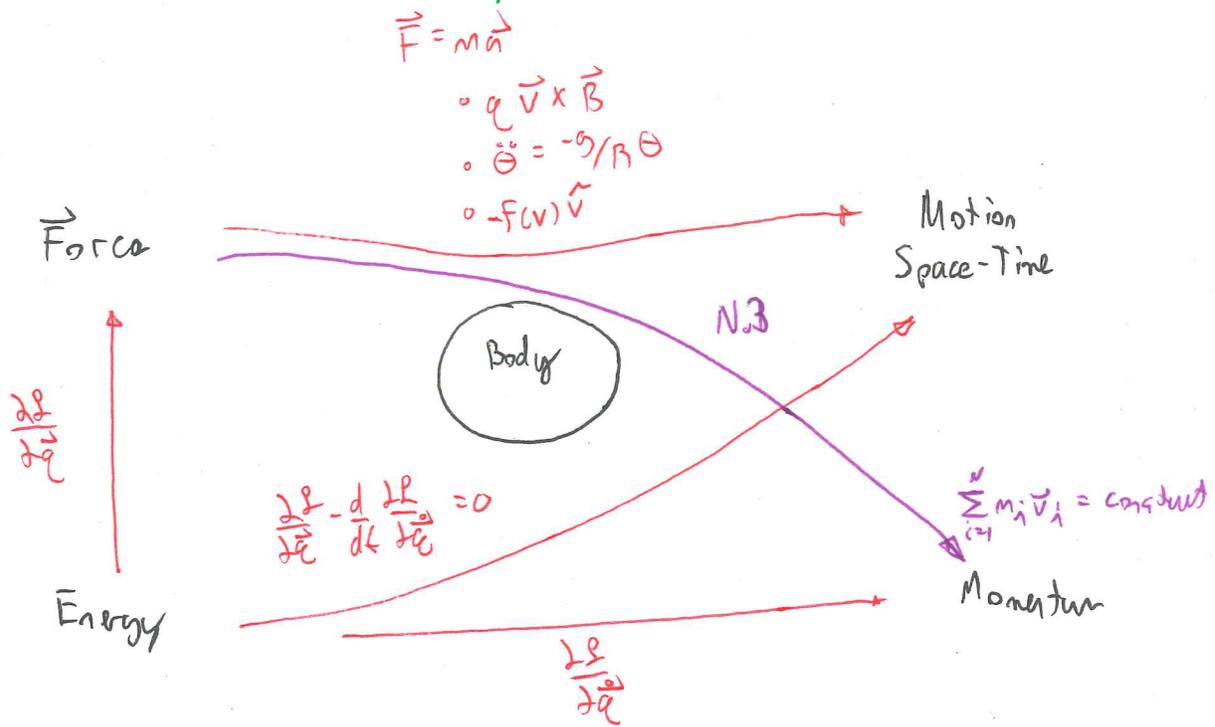


Note:

- Circular Motion CW for $\vec{B} = B\hat{z}$ ϵ $q > 0$
- If $\vec{B} = -B\hat{z}$ then reverse CCW
- If $q < 0$, $\vec{B} = B\hat{z}$ CCW

Physics 411 - Winter 2015

Friday, Week 3 3

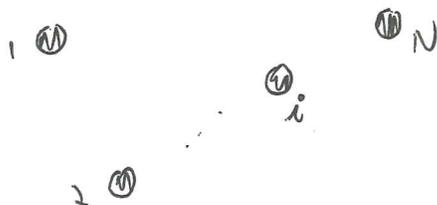


- What does the (nature) of space-time under the Lagrangian Formalism tell us about physical systems?

• Recall,

N.B. \Rightarrow Conservation of momentum

• If $\sum_{i=1}^N \vec{F}_i^{\text{Ext}} = 0$, then $\sum_{i=1}^N m_i \vec{v}_i = \text{constant}$



In the Lagrangian formalism we have:

$$\begin{aligned}
 \mathcal{L} &= T - V \\
 &= \begin{cases} \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}, t) & \text{Single Particle} \\ \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 - \sum_i U_i(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) & \text{System of Particles} \end{cases} \quad (1)
 \end{aligned}$$

Cartesian vector

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} = 0 \quad \text{for } i=1, \dots, N$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_i} = \vec{F}_i \quad [\text{Generalized Force}]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} = \vec{p}_i \quad [\text{Generalized Momentum}]$$

For example, using the single-particle \mathcal{L} from above:

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m \dot{\vec{r}} = m \vec{v}$$

Recall $\vec{r}_i(t), \vec{v}_i(t)$ uniquely specify a system, thus,

$\mathcal{L}(\vec{r}_1, \dot{\vec{r}}_1, \dots, \vec{r}_N, \dot{\vec{r}}_N, t)$ has all information about physical system

∴ \mathcal{L} contains all the physics

Thus, for closed systems, \mathcal{L} cannot change if we (transform) (switch) to a new inertial reference frame.

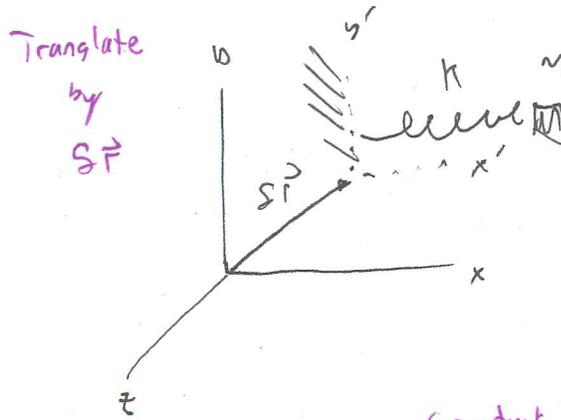
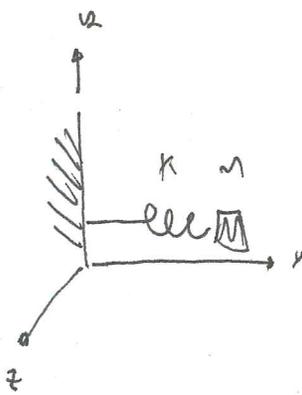
Argues / Proves earlier

So, $\delta \mathcal{L} = 0$ for translations & rotations to new ^{inertial} reference frame

• Emmy Noether: Developed Algebra + symmetry in physics. All symmetries associated w/ a conservation law.

$\delta \mathcal{L} = 0$ for translations (Space is homogeneous & Invariant under translations)

• An example to illustrate $\delta \mathcal{L} = 0$



$$\vec{r}, \quad \dot{\vec{r}} = \vec{v}$$

$$\frac{1}{2} m \vec{v}^2 = \frac{1}{2} m \dot{\vec{r}}^2$$

$$\begin{aligned} \vec{F} &= -k(\vec{r} - \vec{r}_{\text{wall}}) \\ &= -k \Delta \vec{r} \end{aligned}$$

Constant

$$\frac{d}{dt}(\vec{r} + S\vec{r}) = \dot{\vec{r}}$$

$$\frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m \dot{\vec{v}}^2 = \frac{1}{2} m \dot{\vec{r}}^2$$

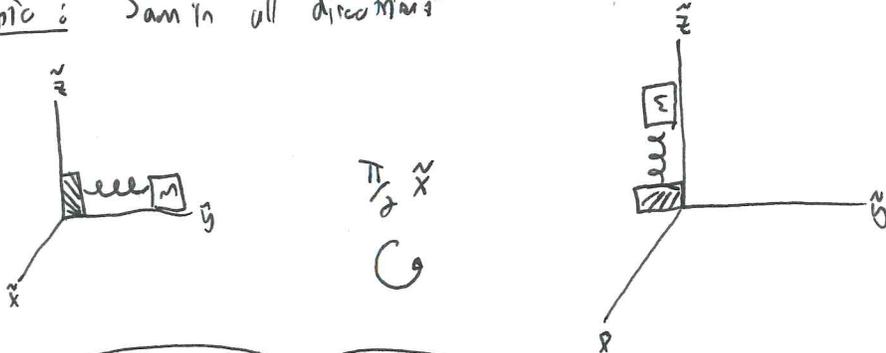
$$\begin{aligned} \vec{F} &= -k((\vec{r} + S\vec{r}) - (\vec{r}_{\text{wall}} + S\vec{r}_{\text{wall}})) \\ &= -k(\vec{r} - \vec{r}_{\text{wall}}) \\ &= -k \Delta \vec{r} \end{aligned}$$

• \mathcal{L} & physics don't change under translations.

$$\begin{aligned}
 \delta L &= \sum_i \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \delta \dot{\mathbf{r}}_i + \frac{\partial L}{\partial \mathbf{r}_i} \cdot \delta \mathbf{r}_i \right) = \sum_i (\mathbf{p}_i + \mathbf{E}_i) \cdot \delta \mathbf{r}_i = \sum_i \mathbf{p}_i \cdot \delta \mathbf{r}_i \\
 &= \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i} \right) \delta \mathbf{r}_i \\
 &= \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \right) \delta \mathbf{r}_i \\
 &= \frac{d}{dt} \left(\sum_i \mathbf{p}_i \right) \delta \mathbf{r}_i = 0 \quad \Rightarrow \quad \sum_i \mathbf{p}_i = \text{constant}
 \end{aligned}$$

Homogeneity of space for a closed system \Rightarrow Conservation of Momentum \Rightarrow N.3

Space is Isotropic: Same in all directions



Make $\delta \vec{\theta}$

$$\delta \vec{a} = \delta \vec{\theta} \times \vec{a}$$

$$\begin{aligned}
 \delta L &= \sum_i \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \delta \dot{\mathbf{r}}_i + \frac{\partial L}{\partial \mathbf{r}_i} \cdot \delta \mathbf{r}_i \right) \\
 &= \sum_i \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot (\delta \vec{\theta} \times \dot{\mathbf{r}}_i) + \frac{\partial L}{\partial \mathbf{r}_i} \cdot (\delta \vec{\theta} \times \mathbf{r}_i) \right) \\
 &= \delta \vec{\theta} \cdot \sum_i \left(\dot{\mathbf{r}}_i \times \frac{\partial L}{\partial \dot{\mathbf{r}}_i} + \mathbf{r}_i \times \frac{\partial L}{\partial \mathbf{r}_i} \right) \\
 &= \delta \vec{\theta} \cdot \sum_i (\dot{\mathbf{r}}_i \times \mathbf{p}_i + \mathbf{r}_i \times \dot{\mathbf{p}}_i) \\
 &= \delta \vec{\theta} \cdot \frac{d}{dt} \left(\sum_i \mathbf{r}_i \times \mathbf{p}_i \right) \\
 &= 0 \quad \Rightarrow \quad \sum_i \mathbf{r}_i \times \mathbf{p}_i = \text{constant}
 \end{aligned}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$\begin{aligned}
 \frac{\partial L}{\partial \dot{\mathbf{r}}_i} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \dot{\mathbf{p}}_i \\
 \frac{\partial L}{\partial \mathbf{r}_i} &= \dot{\mathbf{p}}_i
 \end{aligned}$$

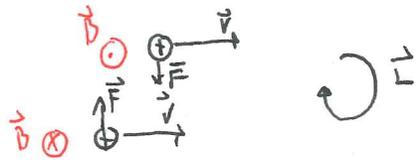
$$\sum_i \mathbf{r}_i \times \mathbf{p}_i = \text{constant}$$

$\vec{L} \equiv \sum_i \mathbf{r}_i \times \mathbf{p}_i$ Angular momentum

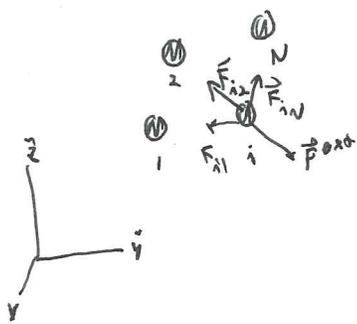
\vec{L} is a property of space for a closed system ~~$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \text{constant}$~~ \Rightarrow

$$\vec{L} = \sum_i \vec{l}_i = \sum_i \vec{r}_i \times \vec{p}_i = \text{constant}$$

\vec{L} for a closed system takes into account all \vec{l}_i , even for EM field; recall



Now define $\vec{l} = \vec{r} \times \vec{p}$ \downarrow consider Newtonian view:



$$\begin{aligned}
 \vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i \\
 \dot{\vec{L}} &= \frac{d}{dt} \left(\sum_i \vec{r}_i \times \vec{p}_i \right) \\
 &= \sum_i \left(\dot{\vec{r}}_i \times m \dot{\vec{v}}_i + \vec{r}_i \times \dot{\vec{p}}_i \right) \\
 &= \sum_{ij} \vec{r}_i \times \vec{F}_{ij} + \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} \quad \left(\vec{F}_{ii} = 0 \right) \\
 &= \frac{1}{2} \left(\sum_{ij} \vec{r}_i \times \vec{F}_{ij} + \sum_{ij} \vec{r}_j \times \vec{F}_{ji} \right) + \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} \\
 &= \frac{1}{2} \left(\sum_{ij} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} \right) + \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} \quad \left(\vec{F}_{ij} = -\vec{F}_{ji} \right)
 \end{aligned}$$

1. If $\vec{F}_{ij} \propto (\vec{r}_i - \vec{r}_j) = \vec{r}_{ij}$, then \vec{F}_{ij} is central force & $\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}}$ N.O.2

2. If $\sum_i \vec{F}_i^{\text{ext}} = 0$ (true for closed system) then $\vec{F}_{ij} \propto \vec{r}_{ij} \Rightarrow \vec{F}_{ij}$ is central.

True if a closed system \longrightarrow

since $\vec{F}_i^{\text{ext}} = 0$

Central force $\vec{F}_G \propto \frac{1}{r^2} \hat{r}$, $\vec{F}_E \propto \frac{1}{r^2} \hat{r}$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{ij} \vec{r}_{ij} \hat{r}_{ij} \times \hat{r}_{ij} F(r_{ij}) + \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} \\
 &= \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} = \sum_i \dot{\vec{L}}_i^{\text{ext}}
 \end{aligned}$$