

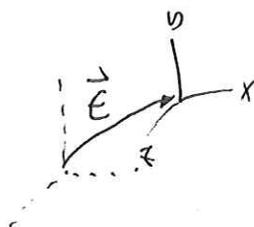
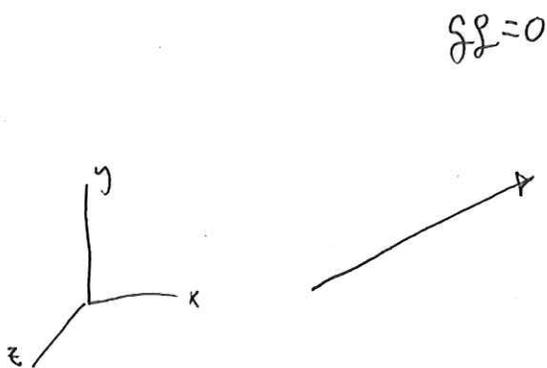
Physics 41, Winter 2015

Monday, Week 43

Review:

$\mathcal{L}(\vec{r}_1, \dot{\vec{r}}_1, \dots, \vec{r}_N, \dot{\vec{r}}_N, t)$ contains all information about evolution of physical system.

In mechanically equivalent (i.e. mutually inertial) ref. frames,

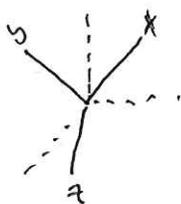


Translational Invariance
Homogeneity of Space

$$\sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial \vec{r}_i} = \sum_{i=1}^N M_i \vec{v}_i = \text{constant}$$

Maybe write as \vec{v}_i ?

$$\Rightarrow \underline{N \cdot 3}; \quad \underline{\vec{F}_{ij} = -\vec{F}_{ji}} \quad \vec{P}_i$$



Rotational Invariance
Isotropy of Space

$$\sum_{i=1}^N \vec{r}_i \times \frac{\partial \mathcal{L}}{\partial \vec{r}_i} = \sum_{i=1}^N \vec{r}_i \times M_i \vec{v}_i = \text{constant}$$

\vec{L}_i

$$\dot{\vec{L}} = \frac{d}{dt} \left(\sum_{i=1}^N \vec{r}_i \times M_i \vec{v}_i \right)$$

$$\sum_{i \neq j}^N (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} + \sum_{i=1}^N \vec{r}_i \times \vec{F}_i^{\text{ext}}$$

If $\vec{F}_{ij} \propto (\vec{r}_i - \vec{r}_j)$, then

$$\dot{\vec{L}} = \sum_{i=1}^N \vec{r}_i \times \vec{F}_i^{\text{ext}} \quad \text{Newton's 2nd Law for rotational motion}$$

$$= \sum_{i=1}^N \vec{r}_i \times \vec{F}_i^{\text{ext}}$$

If system is closed, then

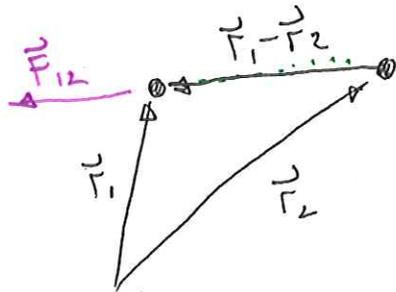
$$\vec{F}_{ij} \propto (\vec{r}_i - \vec{r}_j)$$

i.e. \vec{F}_{ij} is central, like Gravity & Coulomb force.

Ex) Two bodies:

$$\begin{aligned}
 \vec{L} &= \frac{1}{2} \sum_{i,j=1}^2 (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} \\
 &= \frac{1}{2} \left((\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} + (\vec{r}_2 - \vec{r}_1) \times \vec{F}_{21} \right) \\
 &= \frac{1}{2} \left((\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} - (\vec{r}_2 - \vec{r}_1) \times \vec{F}_{12} \right) \\
 &= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} \\
 &= 0 \quad \text{For closed system by Isotropy of Space}
 \end{aligned}$$

↑ $-\vec{F}_{12}$ by Homogeneity of Space



$\Rightarrow \vec{F}_{12}$ acts along $\vec{r}_1 - \vec{r}_2$
i.e. it is central.

So Lagrangian Method & Noether tell us \vec{p} & $\vec{L} = \vec{r} \times \vec{p}$ are conserved quantities.
Let's relate \vec{L} to angular rotational quantities:

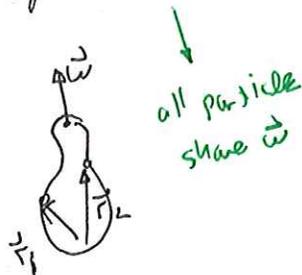
$$\begin{aligned}
 \vec{L} &= \vec{r} \times m\vec{v} \\
 &= m\vec{r} \times (\vec{\omega} \times \vec{r}) \\
 &= m\vec{\omega} (\vec{r} \cdot \vec{r}) - \vec{r} (\vec{r} \cdot \vec{\omega}) \\
 &= m \begin{bmatrix} \omega_x (x^2 + y^2 + z^2) - x(x\omega_x + y\omega_y + z\omega_z) \\ \omega_y (x^2 + y^2 + z^2) - y(x\omega_x + y\omega_y + z\omega_z) \\ \omega_z (x^2 + y^2 + z^2) - z(x\omega_x + y\omega_y + z\omega_z) \end{bmatrix}
 \end{aligned}$$

recall $d\vec{r} = d\vec{\omega} \times \vec{r} \Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$

$$= m \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \vec{\omega}$$

$$= \hat{I} \vec{\omega}$$
 where $I_{ii} \equiv$ moments of inertia
 $I_{ij} \equiv$ products of inertia
 ~~$I \vec{\omega}$~~ in general

For a rigid system of particles,

$$\vec{L} = \sum_{\alpha=1}^N \vec{l}_{\alpha}$$


$$= \hat{I} \vec{\omega}$$

where

$$I_{xx} = \sum_{\alpha=1}^N m_{\alpha} (y_{\alpha}^2 + z_{\alpha}^2) \rightarrow \int \rho(\vec{r}) (y^2 + z^2) dV$$

$$I_{xy} = - \sum_{\alpha=1}^N m_{\alpha} x_{\alpha} y_{\alpha} \rightarrow - \int \rho(\vec{r}) x y dV$$

Punchline later,

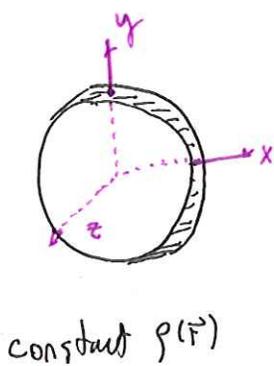
$$\vec{L} = \hat{I} \vec{\omega}$$

$$= I \vec{\omega}$$

↓ scalar

is an eigenvalue problem $\hat{I} \vec{\omega} = I \vec{\omega}$

So only special directions of $\vec{\omega}$ give the simple result.
 Here's an example when it works:



For $\vec{\omega} = \omega \hat{z}$

$$\vec{L} = (I_{xz} \hat{x} + I_{yz} \hat{y} + I_{zz} \hat{z}) \omega$$

$$- \int xz \rho dV \quad - \int yz \rho dV \quad \int (x^2 + y^2) \rho dV$$

odd function integrated over symmetric limits.

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I

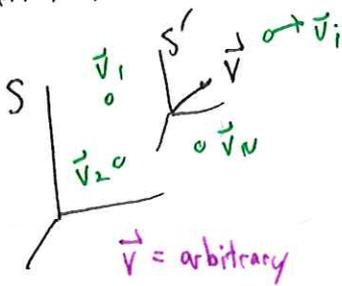
∴

$$\vec{L} = I \omega \hat{z}$$

$$= I \vec{\omega}$$

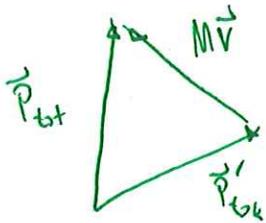
Systems of Particles: Some special properties:

We know $\vec{p} = M\vec{v}$ is a conserved quantity. Let's see how \vec{p} ~~change~~ ^{change} when we transform to a new inertial frame:



then Galilean relativity gives $\vec{v}'_i = \vec{v}_i + \vec{v}$

$$\begin{aligned} \vec{p}_{tot} &= \sum_i m_i \vec{v}'_i \\ &= \sum_{i=1}^N m_i (\vec{v}'_i + \vec{v}) \\ &= \sum_{i=1}^N m_i \vec{v}'_i + \left(\sum_{i=1}^N m_i \right) \vec{v} \end{aligned}$$



$$\boxed{\vec{p}_{tot} = \vec{p}'_{tot} + M\vec{v}}$$

General Transformation Relation

We can adjust \vec{v} so that $\vec{p}'_{tot} = 0$, which give definition of CM velocity:



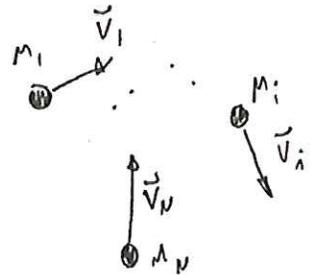
$$\boxed{\vec{v}_{cm} \equiv \frac{1}{M} \sum m_i \vec{v}_i}$$

$\int dt$

$$\boxed{\vec{R}_{cm} \equiv \frac{1}{M} \sum_{i=1}^N m_i \vec{r}_i}$$

Continuous Body

$$\boxed{\vec{R}_{cm} = \frac{1}{M} \int_V \rho(\vec{r}) \vec{r} dV}$$



|||



System of N particles can be treated as a single particle of mass $M = \sum m_i$.

And what about \vec{L} :

$$\begin{aligned} \vec{L} &= \sum \vec{r}_i \times \vec{p}_i \\ &= \sum m_i \vec{r}_i \times (\vec{v}_i' + \vec{v}) \\ &= \sum m_i \vec{r}_i \times \vec{v}_i' + \left(\sum m_i \vec{r}_i \right) \times \vec{v} \end{aligned}$$

$$\downarrow \vec{r}_i' + \vec{r}$$

$$= \sum m_i \vec{r}_i' \times \vec{v}_i' + \vec{r} \times \sum m_i \vec{v}_i' + \left(\sum m_i \vec{r}_i \right) \times \vec{v}$$

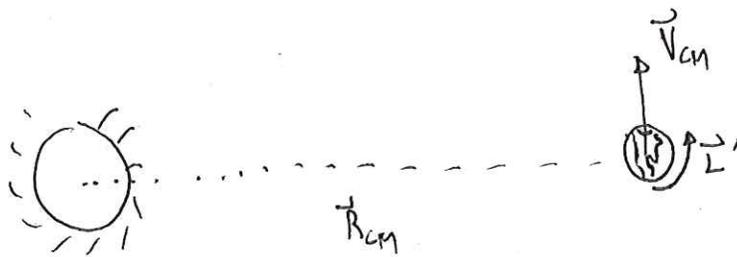
In general

$$\vec{L} = \vec{L}' + M \vec{R}_{CM} \times \vec{V}_{CM}$$

CM Frame

Angular Momentum
in CM frame
"Spin"

Angular Momentum
of CM
"Orbital"



What does invariance of time tell us?

Homogeneity of Time: $\mathcal{L}(\vec{r}_1, \dot{\vec{r}}_1, \dots, \vec{r}_N, \dot{\vec{r}}_N)$

for a closed system, \mathcal{L} cannot depend on time.

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial \vec{r}_i} \dot{\vec{r}}_i + \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \ddot{\vec{r}}_i + \frac{\partial \mathcal{L}}{\partial t} \\ &= \sum_{i=1}^N \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \right) \dot{\vec{r}}_i + \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \ddot{\vec{r}}_i \end{aligned}$$

↓ $\frac{\partial \mathcal{L}}{\partial \vec{r}_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} = 0$

$$\frac{d}{dt} \left(\sum_i \dot{\vec{r}}_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \right)$$

$\mathcal{H} \equiv$ Hamiltonian

or $\frac{d}{dt} \left[\sum_i \dot{\vec{r}}_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} - \mathcal{L} \right] = 0$

$\dot{\mathcal{H}} = \underline{\underline{\text{constant}}}$. But $\mathcal{L} = \sum \frac{1}{2} m_i \dot{\vec{r}}_i^2 - U(r_1, \dots, r_N)$, then

$$\dot{\vec{r}}_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} = m_i \dot{\vec{r}}_i^2$$

$$\dot{\mathcal{H}} = \sum_i \dot{\vec{r}}_i m_i \dot{\vec{r}}_i - \frac{1}{2} \sum m_i \dot{\vec{r}}_i^2 + U(r_1, \dots, r_N)$$

$$= \frac{1}{2} \sum m_i v_i^2 + U$$

Now understand much better...

$$= \text{K.E.} + \text{P.E.}$$

Force

Motion
spacetime

$$= \text{constant} \quad !!!$$

Energy

Momentum

Let's now explore this last conserved quantity, \mathcal{H} , energy, $\dot{\mathcal{H}}$, how it transforms.

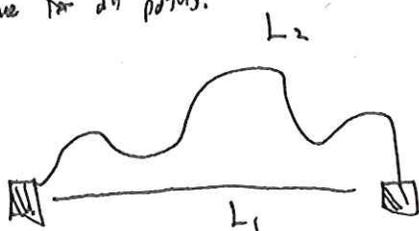
Energy: Recall w/ $\vec{F} = \text{constant}$ w/ $\Delta t \rightarrow dt$:

$$\frac{1}{2} m \vec{V}_f^2 - \frac{1}{2} m \vec{V}_i^2 = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

$$\Delta T \equiv W(\vec{r}_i \rightarrow \vec{r}_f)$$

In general, ΔT is path dependent:

W not same for all paths.



$$W_{L_1} = \int \vec{F} \cdot d\vec{r} = F_f L_1$$

$$W_{L_2} = F_f L_2 \quad W_{L_2} > W_{L_1}$$

Friction
 F_{fr}
↓

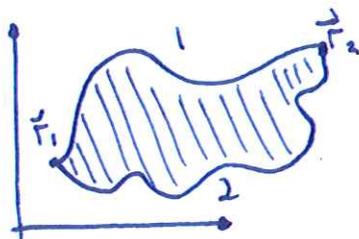
Some forces have work that is path independent, these forces allow us to define a potential energy:
One class of forces we know as conservative forces:

A force is conservative if and only if:

$$\vec{F} = \vec{F}(\vec{r})$$

↳ No \vec{v}, t dependence: $e\vec{v} \times \vec{B}$, $-f(v)\vec{v}$, $\vec{F}_0 \sin(\omega t)$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \text{ is path independent} \quad (\Leftrightarrow) \quad \vec{\nabla} \times \vec{F} = 0$$



$$\int \vec{\nabla} \times \vec{F} = \text{FTC} \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} + \int_{\vec{r}_2}^{\vec{r}_1} \vec{F} \cdot d\vec{r} = 0$$

Stokes
Theorem

$$\Rightarrow \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = - \int_{\vec{r}_2}^{\vec{r}_1} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_2}^{\vec{r}_1} \vec{F} \cdot d\vec{r}$$

Physics 411, Winter 2015

Wednesday, Week 4:

Review

Force

• Conservative force:

$$\textcircled{1} \mathbf{F} = -\nabla\phi$$

$$\textcircled{2} \int \mathbf{F} \cdot d\mathbf{r} \Leftrightarrow \nabla \times \mathbf{F} = 0$$

Body
Mass

Energy

$$H \equiv \sum \dot{\mathbf{r}}_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} - \mathcal{L} \quad [\text{Hamiltonian}]$$

$$\mathcal{L} \equiv T - V$$

Ref. frame where $\vec{p}' = 0$

$$\vec{V}_{cm} = \frac{1}{M} \sum m_i \vec{v}_i$$

$$\vec{R}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i$$

Motion: Space-Time

Momentum:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} = m_i \vec{v}_i = \vec{p}_i$$

$$\begin{aligned} \vec{r}_i \times \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} &= \vec{r}_i \times \vec{p}_i \\ &= \vec{L}_i = \mathbf{I} \vec{\omega}_i \end{aligned}$$

Also, in CM frame

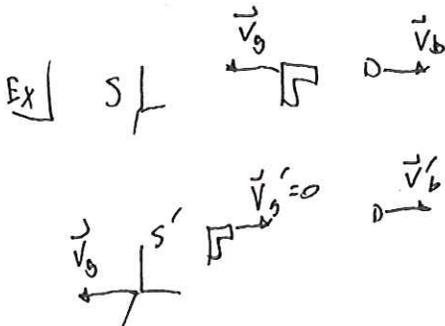
$$\vec{L}_{tot} = \vec{L}'_{tot} + M \vec{R}_{cm} \times \vec{V}_{cm}$$

"Spin" "Orbital"

In defining CM, we encountered \vec{p}_{tot} in two inertial reference frames:

$$\vec{p}_{tot} = \sum m_i \vec{v}_i \quad (1)$$

$$= \sum m_i (\vec{v} + \vec{v}'_i) \quad (2)$$



In frame S,

$$m_a \vec{v}_a + m_b \vec{v}_b = 0$$

$$\vec{v}_b = -\frac{m_a}{m_b} \vec{v}_a$$

In frame S',

$$m_a (\vec{v}_a + \vec{v}) + m_b (\vec{v}_b + \vec{v}) = 0$$

$$\vec{v}_a = -\frac{m_b}{m_a + m_b} \vec{v}_b'$$

$$\text{Then, } \vec{v}_b = \frac{m_a}{m_b} \cdot \frac{m_b}{m_a + m_b} \vec{v}'_b = \frac{1}{1 + m_b/m_a} \vec{v}'_b$$

For \vec{F} conservative,

$$U(\vec{r}) \equiv - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{x}) \cdot d\vec{x} \quad [\text{Potential Energy}]$$

reference point, defines $U(\vec{r})$ & $U(\vec{r}_0) = 0$ usually.

$$= -W(\vec{r}_0 \rightarrow \vec{r})$$

Example of Conservative Forces:

• $\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \cdot \frac{1}{r^2} \hat{r}$

• $\vec{F} = -G \frac{M_1 M_2}{r^2} \hat{r}$

• $\vec{F} = \text{constant}$

Why are they called conservative? Consider $W(\vec{r}_0 \rightarrow \vec{r}_2) = \int_{\vec{r}_0}^{\vec{r}_1} + \int_{\vec{r}_1}^{\vec{r}_2}$

$$= W(\vec{r}_0 \rightarrow \vec{r}_1) + W(\vec{r}_1 \rightarrow \vec{r}_2)$$

$$\Rightarrow W(\vec{r}_0 \rightarrow \vec{r}_2) = W(\vec{r}_0 \rightarrow \vec{r}_2) - W(\vec{r}_0 \rightarrow \vec{r}_1)$$

$$= -U(\vec{r}_2) + U(\vec{r}_1)$$

$$= -(U(\vec{r}_2) - U(\vec{r}_1))$$

$$= -\Delta U$$

$$= \Delta T$$

$$\Rightarrow \Delta T + \Delta U = \Delta(T+U) = 0$$

Can we relate $U(\vec{r}) \rightarrow \vec{F}$? Note,

$$dU(\vec{r}) = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

$$= \nabla U \cdot d\vec{r} \quad [\text{Calculus}]$$

$$= -W(\vec{r} \rightarrow \vec{r} + d\vec{r}) \quad [\text{Def. of Pot. Energy}]$$

$$= -\vec{F}(\vec{r}) \cdot d\vec{r} \quad "$$

$$\circ \circ \quad \vec{F}(\vec{r}) = -\nabla U(\vec{r})$$

We saw this w/ Lagrangian too!!

Notice that $\vec{\nabla} \times \vec{F} = 0 \Rightarrow \int \vec{F} \cdot d\vec{r}$ is path independent so we can define a U such that

$$\vec{F} = -\vec{\nabla}U$$

even if $U(\vec{r}, t)$, so $\vec{\nabla} \times \vec{F} = 0$ alone not good enough. In the case of $U(\vec{r}, t)$,

$$dU = \vec{\nabla}U \cdot d\vec{r} + \frac{\partial U}{\partial t} dt$$

$$= -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t} dt$$

$$\Rightarrow dT = -dU + \frac{\partial U}{\partial t} dt$$

$$\text{or } \boxed{\frac{d}{dt}(T+U) = \frac{\partial U}{\partial t}} \quad \text{so } T+U \text{ not conserved.}$$

Optional or Power Point

Brief math review: $\vec{\nabla}f$; $\vec{\nabla} \times \vec{F}$

$$d\vec{r} = g_1 dr_1 \hat{e}_1 + g_2 dr_2 \hat{e}_2 + g_3 dr_3 \hat{e}_3$$

$$= dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$$

$$\vec{\nabla}f(\vec{r}) = \sum_{i=1}^3 \frac{1}{g_i} \frac{\partial f}{\partial r_i} \hat{e}_i$$

$$\vec{\nabla}f(\vec{r}) = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \hat{\phi}$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{g_1 g_2 g_3} \begin{vmatrix} g_1 \hat{e}_1 & g_2 \hat{e}_2 & g_3 \hat{e}_3 \\ \partial/\partial r_1 & \partial/\partial r_2 & \partial/\partial r_3 \\ g_1 F_1 & g_2 F_2 & g_3 F_3 \end{vmatrix}$$

$$0 = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (r \sin\theta F_\phi) - \frac{\partial F_\theta}{\partial \phi} \right] \hat{r}$$

$$+ \left[\frac{1}{r \sin\theta} \frac{\partial F_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r F_\theta) \right] \hat{\theta}$$

$$+ \left[\frac{1}{r} \left(\frac{\partial}{\partial r} (r F_\theta) - \frac{\partial F_r}{\partial \theta} \right) \right] \hat{\phi}$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r \hat{\theta} & r \sin\theta \hat{\phi} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix} = 0$$

Ex | Conservative force & $U(\vec{r})$:

$$\vec{F}(\vec{r}) = \frac{c}{r^2} \hat{r} \quad \text{No } \vec{v}, t \text{ dependence!}$$

What about $\vec{\nabla} \times \vec{F} = 0$?

In spherical coordinates $F_r = \frac{c}{r^2}$, $F_\theta = F_\phi = 0$

$$\vec{\nabla} \times \vec{F} = \frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \theta} \hat{\phi} - \frac{1}{r} \frac{\partial F_r}{\partial \phi} \hat{\theta}$$

$\downarrow 0$ $\downarrow 0$

$\vec{\nabla} \times \vec{F} = 0$ $\therefore \vec{F} = \frac{c}{r^2} \hat{r}$ is conservative.

$U(\vec{r})$:

$$\begin{aligned} U(\vec{r}) &= - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} = - \int_{r_0}^r \frac{c}{r^2} \hat{r} \cdot (dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}) \\ &= -c \int_{r_0}^r \frac{dr}{r^2} = \frac{c}{r} \Big|_{r_0}^r \\ &= \frac{c}{r} - \frac{c}{r_0} \Big|_{r \rightarrow \infty} \end{aligned}$$

Any path \uparrow

$U(\vec{r}) = \frac{c}{r}$

N.B. $U(\vec{r}) = U(r)$

This is a 1D potential which gives 1D forces, even though system lives in 3D world.

Ex 2-body gravity (Earth-Sun), atom, molecules.

Many important systems are 1D:

Suppose $F = F(x)$

1. Only has x -dependence

2. $W(x_i \rightarrow x_f) = \int_{x_i}^{x_f} F(x) dx$

$$= \int_{x_i}^{x_1} F(x) dx + \int_{x_1}^{x_2} F(x) dx + \dots + \int_{x_N}^{x_f} F(x) dx$$

By calculus,
so explicit path

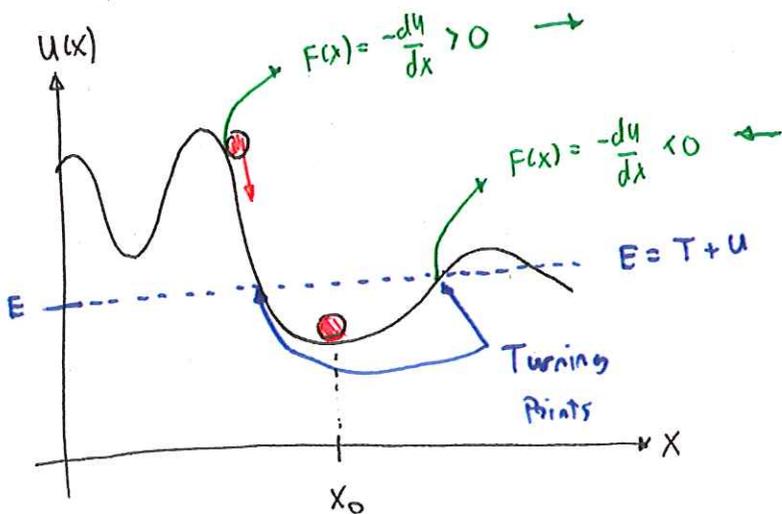
$x_i \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_N \rightarrow x_f$
does change $W(x_i \rightarrow x_f)$

$\circ \circ$ $F(x)$ is conservative!!

Then

$$U(x) = - \int_{x_0}^x F(x) dx \quad F(x) = - \frac{dU(x)}{dx}$$

Energy Landscapes:



$$F(x_0) = 0$$

Taylor Expansion about x_0 $U(x)$:

$$U(x) = U(x_0) + \underbrace{U'(x_0)}_0 (x-x_0) + \frac{1}{2} U''(x_0) (x-x_0)^2 + \dots$$

so

$$U(x) \approx U(x_0) + \frac{1}{2} U''(x_0) (x-x_0)^2$$

$U''(x_0) > 0$ Stable equilibrium

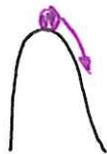
$U''(x_0) < 0$ Unstable equilibrium

$$F(x) = -\frac{dU}{dx}$$

$$= -U''(x_0)(x - x_0)$$



$$U''(x_0) > 0$$



$$U''(x_0) < 0$$

$$F(x) = -k \Delta x$$

$$U(x) = \frac{1}{2} k \Delta x^2$$

Motion about stable equilibrium is described by a Hookean force.

One-Dimensional systems, we have another way of ascertaining motion from energy:

By conservation of energy,

$$E = T + U$$

$$= \frac{1}{2} m \dot{x}^2 + U(x)$$

Solve for \dot{x} :

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U(x))}$$

so

$$\int_{x_0}^x \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = \int dt$$

Ex) $U(x) = \frac{1}{2} k x^2$

~~with~~ $s = \sqrt{\frac{1}{2} \frac{k}{E}} x$; ~~with~~ $dx = \sqrt{\frac{2E}{k}} ds$

$$t = \int \frac{dx}{\sqrt{\frac{2}{m}(E - \frac{1}{2} k x^2)}} = \int \frac{\sqrt{\frac{2E}{k}} ds}{\sqrt{\frac{2E}{m}(1 - s^2)}} = \sqrt{\frac{m}{k}} \int \frac{ds}{(1 - s^2)^{1/2}}$$

$s = \sin \theta$
 $ds = \cos \theta d\theta$

$$= \sqrt{\frac{m}{k}} \int \frac{\cos \theta d\theta}{\sqrt{1 - \sin^2 \theta}} = \sqrt{\frac{m}{k}} \theta = \sqrt{\frac{m}{k}} \sin^{-1} \left(\sqrt{\frac{k}{2E}} \cdot x \right) \Big|_{x_0}^x$$

$$x(t) = \sqrt{\frac{2E}{k}} \sin \left(\sqrt{\frac{k}{m}} \cdot t + \theta \right)$$

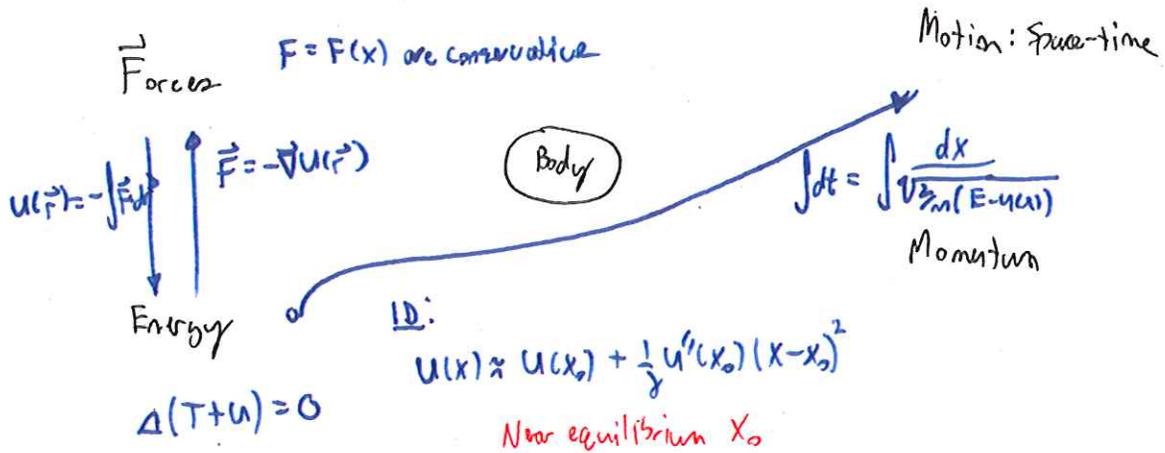
with $E = \frac{1}{2} k x_0^2 \Rightarrow$

$$x(t) = x_0 \sin(\omega t + \theta)$$

Physics 411, Winter 2015

Friday, Week 4

Review 8



Example:

$$U(x) = \frac{1}{2} K x^2$$

$$s \equiv \sqrt{\frac{1}{2} \frac{K}{E}} x \quad dx = \sqrt{\frac{2E}{K}} ds$$

$$t = \int \frac{dx}{\sqrt{\frac{2}{m} (E - \frac{1}{2} K x^2)}} = \int \frac{\sqrt{\frac{2E}{K}} ds}{\sqrt{\frac{2E}{m} (1 - s^2)^{1/2}}} \quad \begin{matrix} s \equiv \sin \theta \\ ds = \cos \theta d\theta \end{matrix}$$

$$= \sqrt{\frac{m}{K}} \int \frac{\cos \theta d\theta}{\sqrt{1 - \sin^2 \theta}} = \sqrt{\frac{m}{K}} \theta = \sqrt{\frac{m}{K}} \sin^{-1} \left(\sqrt{\frac{K}{2E}} x \right) \Big|_{x_0}^x$$

$$x(t) = \sqrt{\frac{2E}{K}} \sin \left(\sqrt{\frac{K}{m}} t + s \right)$$

w/ $E = \frac{1}{2} K x_0^2$

$$x(t) = x_0 \sin(\omega t + s)$$

A statement about rotationally invariant 1D forces $\{$ conservative forces:

Consider a closed system, \rightarrow already know $\vec{F}(\vec{r}) = F(r)\hat{r} = F(r, \theta, \phi)\hat{r}$
i.e. central forces

First with $\vec{F}(\vec{r})$ rotationally invariant (i.e. no ϕ, θ depend)

then $\vec{F}(\vec{r}) = F(r)\hat{r}$
 \downarrow
 $r = |\vec{r}|$

$$\nabla \times \vec{F} = \frac{1}{r \sin \theta} \frac{\partial F(r)}{\partial \phi} \hat{\phi} - \frac{1}{r} \frac{\partial F(r)}{\partial \theta} \hat{\theta}$$

$= 0 \quad \therefore \vec{F}(\vec{r})$ is conservative.

Second, with $\vec{F}(\vec{r})$ conservative, then

$$\begin{aligned} \vec{F}(\vec{r}) &= -\vec{\nabla} U(\vec{r}) \\ &= -\left(\frac{\partial U(\vec{r})}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial U(\vec{r})}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial U(\vec{r})}{\partial \phi} \hat{\phi} \right) \\ &= -\frac{\partial U(\vec{r})}{\partial r} \hat{r} \quad (\text{Because forces must be central}) \end{aligned}$$

Thus $\frac{\partial U(r, \theta, \phi)}{\partial \theta} = \frac{\partial U}{\partial \phi} = 0 \Rightarrow$ rotationally invariant $\vec{F}(\vec{r}) = \vec{F}(r)$

Then for a closed system:

$$\vec{F}(\vec{r}) = F(r)\hat{r} \quad \text{iff} \quad \vec{F}(\vec{r}) \text{ is conservative}$$

\downarrow
1D

$\Rightarrow \exists U(\vec{r}) = U(r)$

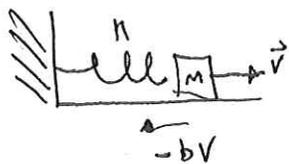
So 1D problems are important (and ubiquitous) as are Hookean potentials:

$$U(x) = \frac{1}{2} kx^2 \quad \text{w/} \quad F(x) = -\frac{dU}{dx}$$

$$m\ddot{x} = -kx \quad \text{w/ solution}$$

$$x(t) = X_0 \cos(\omega t + \phi)$$

In reality, we'll have damping as well, so



O_2 molecule stretching

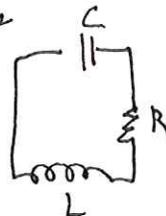
$$m\ddot{x} = -kx - b\dot{x}$$

[Damped SHO]

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

same as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad (1)$$



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$

What's the solution? (It depends...)

$$x(t) = C e^{\lambda t} \quad \text{in (1) since}$$

$$C \lambda^2 e^{\lambda t} + 2\beta \lambda C e^{\lambda t} + \omega_0^2 C e^{\lambda t} = 0$$

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0$$

$$\lambda_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

So general solution is

$$x(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}$$

Exact form of $x(t)$ depends on relationship between β & ω_0 .

$$\beta = 0, \quad \beta < \omega_0, \quad \beta = \omega_0, \quad \beta > \omega_0$$

Undamped ($\beta=0$):

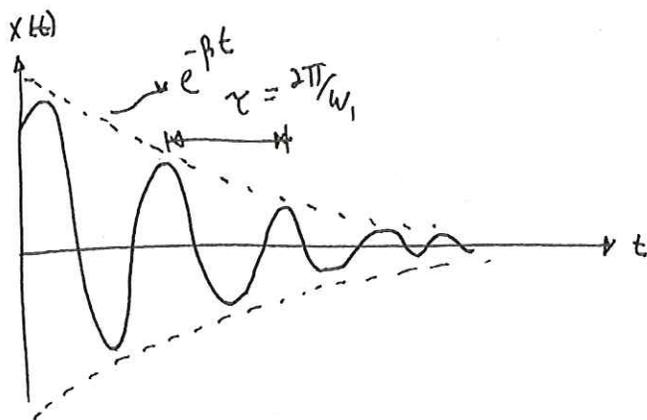
$$\lambda_{1,2} = \pm i\omega_0 \quad x(t) = A \cos(\omega_0 t + S)$$

Weaker Damping ($\beta < \omega_0$):

$$\lambda_{1,2} = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$
$$= -\beta \pm i\omega_1$$

$$x(t) = c_1 e^{-\beta t} e^{i\omega_1 t} + c_2 e^{-\beta t} e^{-i\omega_1 t} \quad c_1, c_2 \in \mathbb{C}$$

$$= \boxed{A e^{-\beta t} \cos(\omega_1 t + S)}$$



β = decay parameter

in $t = \frac{1}{\beta}$, $x(t)$ decreases by $1/e$.

Consider $\frac{x(t)}{\Delta x(t)_{\text{cycle}}} = \frac{x(t)}{x(t) - x(t+\tau)} = \frac{e^{-\beta t}}{e^{-\beta t} - e^{-\beta(t+\tau)}}$

$$= \frac{1}{1 - e^{-\beta\tau}} = \frac{1}{1 - e^{-2\pi\beta/\omega_1}}$$

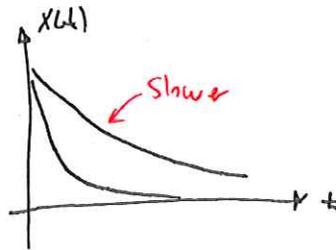
If $\omega_0 \gg \beta$, $\approx \frac{1}{1 - e^{-2\pi\beta/\omega_0}} = \frac{1}{1 - (1 - 2\pi\beta/\omega_0)} = \frac{\omega_0}{2\pi\beta} = \frac{Q}{\pi}$

$$Q \equiv \frac{\omega_0}{2\beta}$$

Quality Factor; large $Q \Rightarrow$ weaker damping

Strongly Damping ($\beta > \omega_0$):

$$x(t) = \underbrace{C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t}}_{\text{Slower}} + \underbrace{C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}}_{\text{Faster}}$$



Critical Damping ($\beta = \omega_0$):

Here, $x(t) = C_1 e^{-\beta t} + C_2 e^{-\beta t}$

↓ ↓
linearly dependent

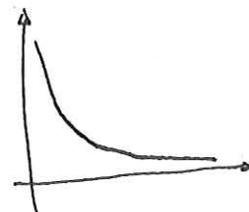
so only one $e^{-\beta t}$ is allowed. Say $x(t) = A e^{-\beta t}$. Now take $\frac{\partial}{\partial \beta}$ of EOM

$$\frac{\partial}{\partial \beta} \left(\frac{d^2 x}{dt^2} + 2\beta \frac{dx}{dt} + \beta^2 x \right) = 0$$

$$\frac{d^2}{dt^2} \left(\frac{\partial x}{\partial \beta} \right) + \underbrace{2 \frac{dx}{dt}}_{\substack{\downarrow 0 \\ -2\beta x}} + 2\beta \frac{d}{dt} \left(\frac{\partial x}{\partial \beta} \right) + \underbrace{2\beta x}_{\downarrow 0} + \beta^2 \frac{\partial x}{\partial \beta} = 0$$

$\Rightarrow \frac{\partial x}{\partial \beta}$ is a solution too!!

$$\frac{\partial x}{\partial \beta} = -A t e^{-\beta t}$$



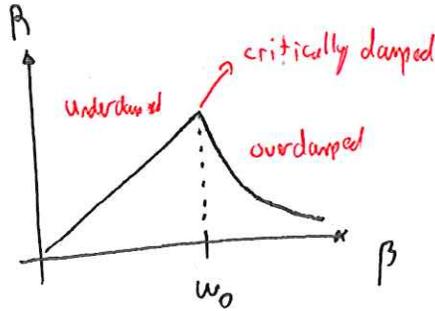
\Rightarrow $x(t) = A_1 e^{-\beta t} + A_2 t e^{-\beta t}$

In summary,

they all have form

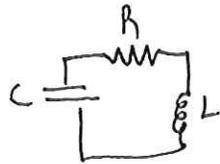
$$X(t) \propto e^{-\beta t}$$

Decay parameter



So if you want to engineer a system to damp as quickly as possible, set $\omega_0 = \beta$.

Project?

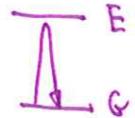
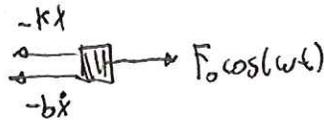


$$\ddot{Q} + \frac{R}{L} \dot{Q} + \frac{1}{LC} Q = 0$$

$$\beta = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Measure $Q(\omega)$ obtain this curve

Driven-damped SHOs



The EoM is given by $F=ma$

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m} \cos(\omega t)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t} \quad (1)$$

Again we guess:

$$X(t) = Ae^{i\omega t}$$

$$f_0 \in \mathbb{R}$$

$$A \in \mathbb{C}$$