

# Physics 411 - Winter 2015

Wednesday, Week 6 3

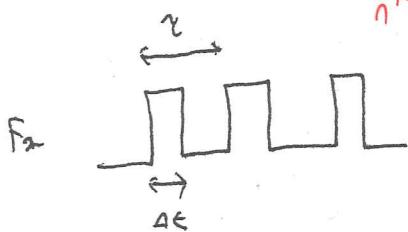
- Proposals due this Friday. ~ 30 hrs of work for project!

- Review:

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f(t)$$

$$= \sum a_n \cos(\omega_n t) + \sum b_n \sin(\omega_n t)$$

$n^{th}$  term has frequency  $\omega_n$  where  $\omega = \frac{2\pi}{T}$



we found:

$$x(t) = \sum A_n(\omega) \cos(\omega_n t + \phi_n(\omega))$$

$$\frac{a_n}{\sqrt{(w_0^2 - w_n^2)^2 + 4\beta^2 w_n^2}}$$

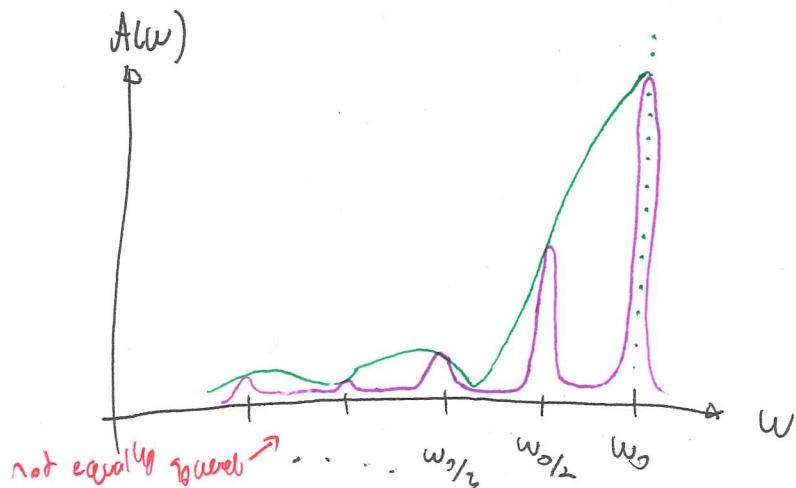
determined by Fourier coeff.

$$\tan^{-1} \left( \frac{-2\beta w_n}{w_0^2 - n^2 w^2} \right)$$

Get resonance when  $w_0 \approx \omega_n$  or when

$$\boxed{w = w_0/n}$$

Height of resonance set by  $\underline{a_n}$ .



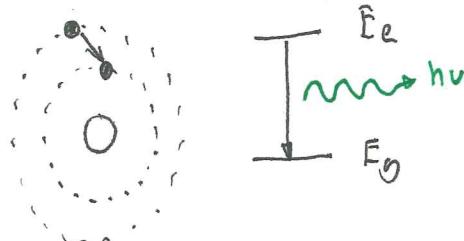
①

Also saw a relation between the width of the pulse in time  $\Delta t$ , the spread in frequencies required to reproduce waveform:

$$\left. \begin{array}{l} \Delta w \Delta t = 2\pi \\ \Delta v \Delta t = 1 \end{array} \right\} \text{w.t}$$

$$\left. \begin{array}{l} \Delta k \Delta x = 2\pi \\ \Delta p \Delta x = h \end{array} \right\} \text{n.x}$$

(Ex)



If emission takes place in a time  $\Delta t$ , then

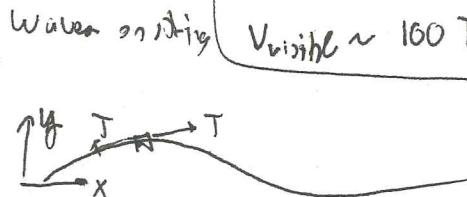
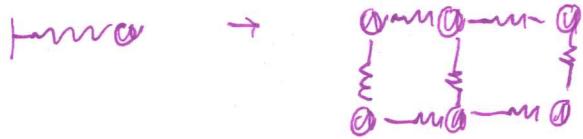
$$\Delta v = \frac{1}{\Delta t} \gg 1$$

$$\Delta t \sim 1 \text{ ns} = 10^{-9} \text{ s}$$

$$\Delta v \sim 1 \text{ GHz}$$

$v_{visible} \sim 100 \text{ THz}$  so not big shift  
but clear limit!!

### Oscillations in rotating systems:



$$\sum F_x = T \cos(\theta(x+\Delta x)) - T \cos \theta(x)$$

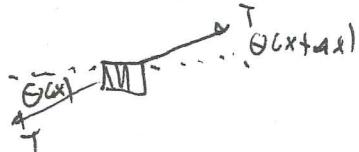
$$\approx T - T \\ = 0$$

$$\sum F_y = m \ddot{y} = \Delta x \mu \ddot{y}$$

$$= T \sin \theta(x+\Delta x) - T \sin \theta(x)$$

$$= T \left( \frac{\partial \theta}{\partial x} \Big|_{x+\Delta x} - \frac{\partial \theta}{\partial x} \Big|_x \right)$$

$$\therefore \ddot{y} = \frac{T}{m} \left. \frac{\partial \theta}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial \theta}{\partial x} \right|_x$$



$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{m} \frac{\partial^2 \theta}{\partial t^2}$$

Wave Equation

Q:

What is the solution to

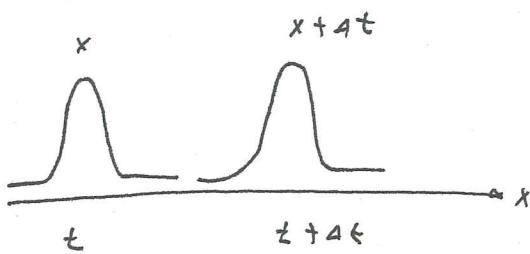
$$\frac{\frac{d^2y}{dt^2}}{L^2} = c^2 \frac{\frac{d^2y}{dx^2}}{L^2}$$

How do we interpret  $c$ ? why?

Any function  $f = f(x \pm ct)$  solves the wave equation.

$$y = \exp\left(\frac{(x+ct)^2}{2}\right) \text{ or } y = \sin(f(x-ct))$$

And  $c$ ?



$$y(x-ct) = y(x+4x - c(t+4t))$$

$$\Rightarrow x-ct = x+4x - c(t+4t)$$

$$\Rightarrow \frac{dx}{dt} = c \quad [\text{Velocity}]$$

- → right travelling  
+ → left travelling

Q: how can we use what we know to determine:

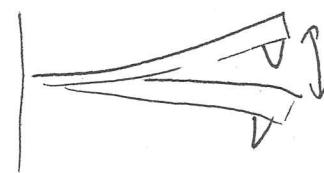
AFM?

↑ Show  
POT



+ strain/stress  
shape

Deflection



Vibration

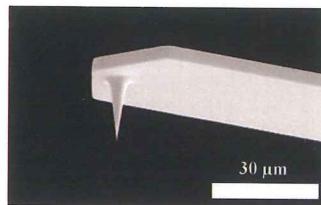
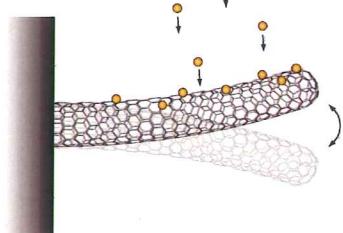
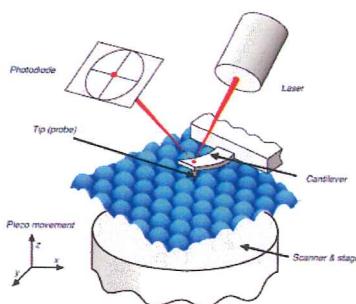
Find shape of beam under the influence of external load & its own weight.

③

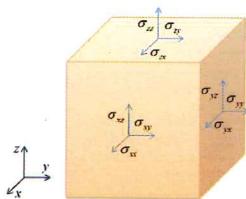
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Friday, Week 6

## Motion and Deflection of Beams:



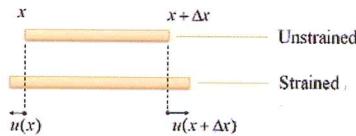
### Stress:



Differential Volume Element for Stress Tensor

$$\sigma = F/A$$

### Strain:



Displacement in x direction at position x

$$\epsilon_{xx} \equiv \frac{u_x(x + \Delta x) - u_x(x)}{\Delta x} = \frac{\partial u_x}{\partial x}$$

$$\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Negative strain = compression  
Positive strain = tension

Generalized Hooke's Law

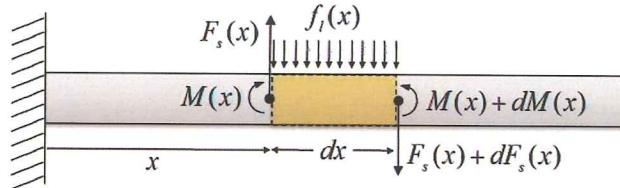
$$\sigma_{ii} = E\epsilon_i \quad \sigma_{ij} = G\epsilon_{ij}$$

$$\sigma_{ij} = \sigma_j i$$

$$F = \frac{A}{L} E \Delta L \quad k = AE/L$$

## Static Deformation of Beams:

Torques, shear forces, and external forces (loads)



Balancing Forces (N.2):

$$f_l(x)dx + (F_s + dF_s) - F_s = 0$$

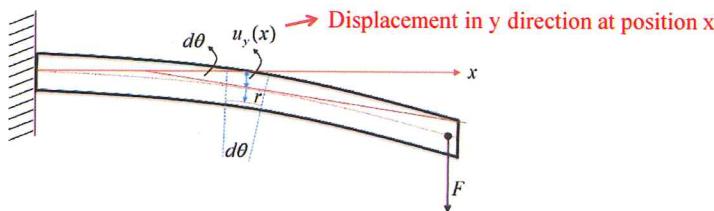
$$\frac{dF_s}{dx} = -f_l(x)$$

Balancing Torques (N.θ. 2) :

$$(M(x) + dM(x)) - M(x) - (F_s(x) + dF_s(x))dx - \frac{f_l(x)dx}{2}dx = 0$$

$$\frac{dM(x)}{dx} = F_s(x)$$

## The stress-strain relationship:



Length of  $dx$   
Neutral Axis  
 $dx = R(x)d\theta$

$$\begin{aligned} \text{Length of } dx \text{ a distance} \\ r \text{ from neutral axis} \\ dl = (R(x) - r)d\theta \\ dl = dx - \frac{r}{R(x)}dx \end{aligned}$$

$$\begin{aligned} \rightarrow & \text{Strain on element} \\ \epsilon_x &= \frac{dl - dx}{dx} \\ &= -\frac{r}{R(x)} \end{aligned}$$

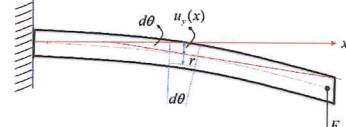
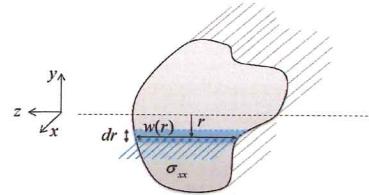
Stress-Strain Relationship

$$\sigma_{xx} = -\frac{r}{R(x)}E$$

$$\frac{dF_s}{dx} = -f_l(x)$$

$$\frac{dM(x)}{dx} = F_s(x)$$

### Torque-stress, and the EOM for $u_y(x)$ :



$$\frac{1}{R(x)} = \frac{\frac{d^2 u_y(x)}{dx^2}}{\left(1 + \left(\frac{du_y(x)}{dx}\right)^2\right)^{3/2}}$$

$$\frac{du_y(x)}{dx} \ll 1$$

$$\frac{1}{R(x)} = \frac{d^2 u_y(x)}{dx^2}$$

We can now use  $\sigma_{xx}$  to obtain the internal bending moment.

$$\sigma_{xx} = -\frac{r}{R(x)} E$$

$$M_z = \int \sigma_{xx} w(r) r dr$$

$$= \int \frac{-E}{R(x)} w(r) r^2 dr$$

$$= -\frac{E}{R(x)} \int w(r) r^2 dr$$

$$= -\frac{E}{R(x)} I$$

Euler Beam Equation

$$\frac{d^2 u_y(x)}{dx^2} = \frac{-M(x)}{EI}$$

differentiate  $\boxed{\frac{dM(x)}{dx} = F_s(x)}$

$$\frac{d^3 u_y(x)}{dx^3} = \frac{-F_s(x)}{EI}$$

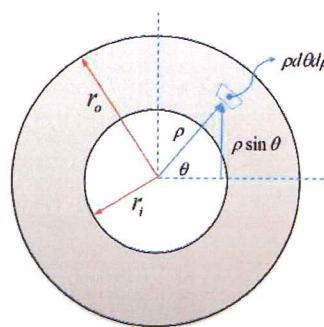
differentiate  $\boxed{\frac{dF_s}{dx} = -f_l(x)}$

$$EI \frac{d^4 u_y(x)}{dx^4} = f_l(x)$$

### An example of the areal moment of inertia:

Areal Moment of Inertia

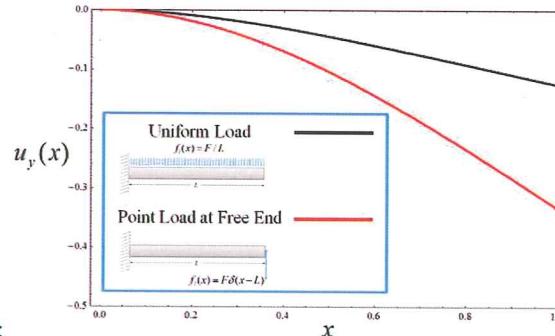
$$I = \int r^2 dA$$



$$\begin{aligned} I &= \int_{r_i}^{r_o} \int_0^{2\pi} \rho^2 \sin^2(\theta) (\rho d\theta d\rho) \\ &= \int_{r_i}^{r_o} \int_0^{2\pi} \rho^3 \left( \frac{1}{2} - \frac{1}{2} \cos(2\theta) \right) d\theta d\rho \\ &= \pi \frac{r_o^4 - r_i^4}{4} \end{aligned}$$

### $u_y(x)$ for uniform and point loads

$$EI \frac{d^4 u_y(x)}{dx^4} = f_l(x)$$



Point load at free end of cantilever:

$$u_y(x) = \frac{F}{EI} \left( \frac{1}{6}x^3 - \frac{L}{2}x^2 \right)$$

$$F = -\frac{EI}{3L^3} u_y(L)$$

Uniform Load:

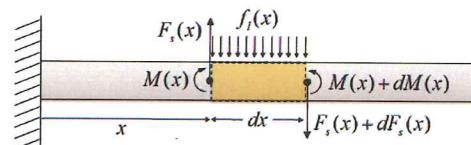
$$u_y(x) = -\frac{F}{24EI} x^4 + \frac{F}{6EI} x^3 - \frac{FL}{4EI} x^2$$

$$k_{eff} = \frac{3EI}{L^3}$$

$$k_{eff} = \frac{8EI}{L^3}$$

### Flexural Vibrations of Beams:

Find EOM for  $u_y(x, t)$ :



Forces (no  $f_l(x)$ ):

$$F_s(x) - F_s(x) - dF_s(x) - \rho A dx \frac{\partial^2 u_y(x)}{\partial t^2} = 0$$

Torques:

$$(F_s(x) + dF_s(x))dx + (M(x) + dM(x)) - M(x) = 0$$

$$\begin{aligned} \frac{\partial F_s}{\partial x} &= \rho A \frac{\partial^2 u_y}{\partial t^2} \\ F_s(x) &= -\frac{\partial M(x)}{\partial x} \end{aligned}$$

$$\frac{d^2 u_y(x)}{dx^2} = \frac{-M(x)}{EI}$$

$$EI \frac{\partial^4 u(x)}{\partial x^4} = -\rho A \frac{\partial^2 u(x)}{\partial t^2}$$

Wave Equation for Beams

Find  $u_y(x, t)$  for cantilever:

Trial Solution:

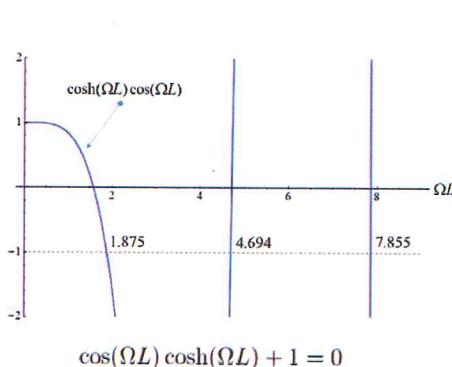
$$EI \frac{\partial^4 u(x)}{\partial x^4} = -\rho A \frac{\partial^2 u(x)}{\partial t^2} \quad u(x, t) = e^{i(kx - \omega t)}$$

$$k^4 = \left(\frac{\rho A}{EI}\right) \omega^2 \quad k = \pm i \left(\frac{\rho A}{EI}\right)^{1/4} \omega^{1/2} \quad \Omega = \left(\frac{\rho A}{EI}\right)^{1/4} \omega^{1/2}$$

$$u(x, t) = e^{-i\omega t} (Ae^{-i\Omega x} + Be^{+i\Omega x} + Ce^{\Omega x} + De^{-\Omega x})$$

$$u(x) = a \cosh(\Omega x) + b \sinh(\Omega x) + c \cos(\Omega x) + d \sin(\Omega x)$$

$$\begin{aligned} \left. \begin{aligned} u(0) &= 0 \\ \frac{du(0)}{dx} &= 0 \end{aligned} \right\} & a = -c \text{ and } b = -d \\ \left. \begin{aligned} \frac{d^2 u(L)}{dx^2} &= 0 \\ \frac{d^3 u(L)}{dx^3} &= 0 \end{aligned} \right\} & \begin{pmatrix} \cosh(\Omega L) + \cos(\Omega L) & \sinh(\Omega L) + \sin(\Omega L) \\ \sinh(\Omega L) - \sin(\Omega L) & \cosh(\Omega L) + \cos(\Omega L) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \\ & \downarrow \\ & \cos(\Omega L) \cosh(\Omega L) + 1 = 0 \end{aligned}$$

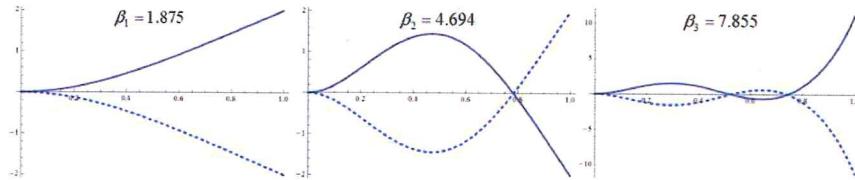


$$\Omega_n L = 1.875, 4.694, 7.855, \quad a_n/b_n = -1.3622, -0.9819, -1.008, \text{ and so on.}$$

$$u_n(x) = a_n [\cosh(\Omega_n x) - \cos(\Omega_n x)] + b_n [\sinh(\Omega_n x) - \sin(\Omega_n x)]$$

## Flexural Mode Shapes:

$$u_n(x) = a_n[\cosh(\Omega_n x) - \cos(\Omega_n x)] + b_n[\sinh(\Omega_n x) - \sin(\Omega_n x)]$$



$$\begin{aligned}\omega_n &= \left(\frac{EI}{\rho A}\right)^{1/2} \Omega_n^2 \\ &= \left(\frac{EI}{\rho A}\right)^{1/2} \frac{\beta_n^2}{L^2} \\ &= \left(\frac{EI}{m L^3}\right)^{1/2} \beta_n^2\end{aligned}$$

$\omega_1$	1
$\omega_2$	6.267
$\omega_3$	17.68
$\omega_4$	34.53

$$k = EI/L^3 \quad \omega = \sqrt{k/m}$$

