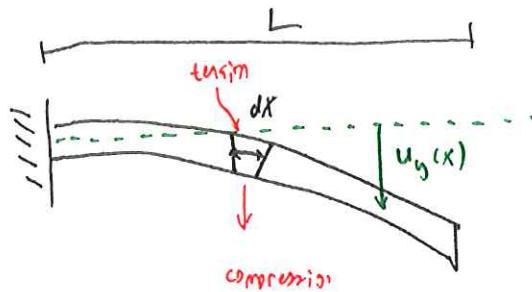


Physics 411 - Winter 2015

Monday, week 7

Review: Stress-Strain relationship:

$$\sigma_{ii} = E \epsilon_{ii} \quad \sigma_{ij} = G \epsilon_{ij}$$



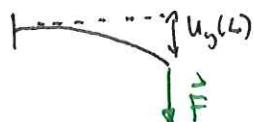
Use:
 $\vec{\sigma} \propto \vec{\epsilon}$

$$\vec{F} = m\vec{a}$$

$$\vec{\tau} = I\vec{\alpha}$$

Static case:

$$EI \frac{d^4 u_y(x)}{dx^4} = f_x(x) \quad \text{Euler-Bernoulli Eq.}$$



$$F = -K u_y(L)$$

where $K = \frac{3EI}{L^3}$

Dynamic Case:

$$EI \frac{d^4 u_y(x)}{dx^4} = -\rho A \frac{d^2 u(x)}{dt^2}$$

Young's Modulus

$$\int r^2 dA$$

Area density

wave E_p.
for beams

where

$$\beta_1 = 1.875$$

$$\beta_2 = 4.694$$

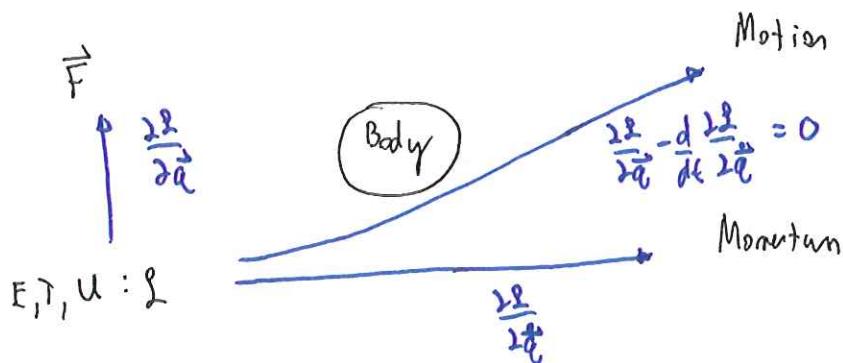
$$\beta_3 = 7.155$$

$$\omega_n = \left(\frac{EI}{mL^3} \right)^{1/2} \beta_n^2$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$k = \left(\frac{EI}{L^3} \right)^{1/2}$$

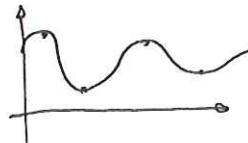
The calculus of variations:



Where does Euler-Lagrange Eq. come from?

In differential calculus, we find the extremum values of $f(x)$ by

$$\frac{df(x)}{dx} = 0$$



Here, $f(x)$ is a function of the variable x .

What if we have some function that is a function of functions, how do we find extremum values?

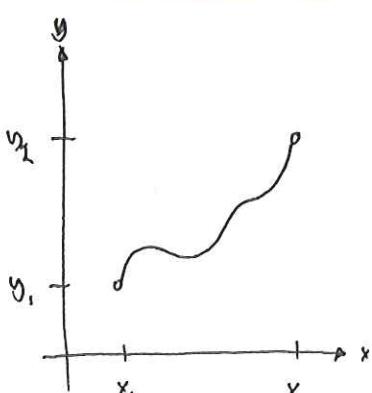
Namely, what if:

$$S = \int f(y(x), y'(x), x) dx$$

How do we find $y(x)$ that minimizes S ?

Q: Are S 's common?
↓, minimizing S common

Find path length for curve in 2D:



$$\text{Say } \vec{r}(t) = (x(t), y(t))$$

$$\begin{aligned} M &= \int \vec{r}(t) dt \\ &= \sqrt{\dot{x}^2 + \dot{y}^2} dt \quad \rightarrow L(\dot{x}, \dot{y}, t) = \int \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned}$$

$$\begin{aligned} &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \rightarrow L(y, y', x) = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \quad (2) \end{aligned}$$

How do we find path that minimizes L ? We need to test all $\{y(x)\}$.

$$\int_{t_1}^{t_2} \sqrt{dx^2 + dy^2} dt$$

Another example:

Fermat's Principle: Path travelled by light is that which minimizes the time of travel.

$$t = \int dt = \int \frac{n(x,y)}{c} ds$$

$$= \frac{1}{c} \int n(x,y) \sqrt{1+y'(x)^2} dx \quad \Rightarrow \quad t = t(x, y, y')$$

Which $y(x)$ will minimize this?

In the Lagrangian formalism, there is some function \mathcal{L} that contains all the information regarding motion. Since $\vec{r}_i(t), \dot{\vec{r}}_i(t)$ enough to recreate path \vec{s} , net forces (\vec{F}) etc., then

$$\mathcal{L} = \mathcal{L}(\vec{r}_1, \dot{\vec{r}}_1, \dots, \vec{r}_N, \dot{\vec{r}}_N, t)$$

for N particles. The actual path of the system is one such that :

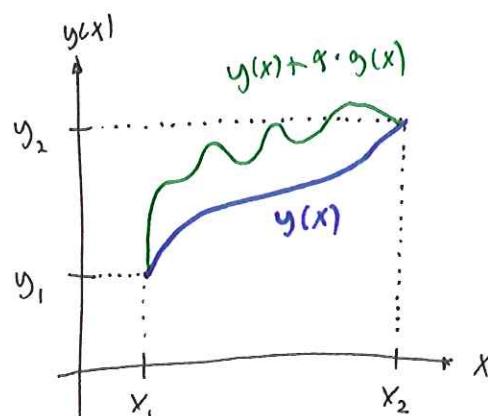
$$S = \int_{t_1}^{t_2} \mathcal{L} dt \quad [\text{Action}]$$

is a minimum. So how do we calculate the minimum? How do we find the functions $x(t), y(t), x'(t), y'(t)$ that minimize S ?

The Calculus of Variations:

Consider

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx$$



and assume $y(x)$ minimizes S .

Now let $g(x)$ be any function satisfying $g(x_1) = g(x_2) = 0$ $\forall \alpha \in \mathbb{R}^+$ consider

$$S(\alpha) = \int_{x_1}^{x_2} f\left(\frac{w(x)}{y+\alpha g}, \frac{w'(x)}{y'+\alpha g'}, x\right) dx$$

(3)

(4)

We know $y(x)$ is the actual path so $S(\alpha)$ is a minimum when $\dot{\alpha} = 0$ & $\frac{dS}{d\alpha} = 0$

Let's calculate $\frac{dS}{d\alpha}$:

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial w} \frac{\partial w}{\partial \alpha} + \frac{\partial F}{\partial w'} \frac{\partial w'}{\partial \alpha} \right) dx$$

Note: $\frac{\partial w}{\partial y} = 1$ $\frac{\partial w'}{\partial y'} = 1$

$$\partial w = \partial y$$

$$\frac{\partial w'}{\partial y'} = 1$$

$$\partial w' = \partial y'$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} g + \frac{\partial F}{\partial y'} g' \right) dx$$

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} g \right) = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) g + \frac{\partial F}{\partial y'} \frac{dg}{dx}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} g - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) g \right) dx +$$

$$\int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} g \right) dx$$

$$\frac{\partial F(x_2)}{\partial y'} g(x_2) - \frac{\partial F(x_1)}{\partial y'} g(x_1)$$

$$= \int_{x_1}^{x_2} g(x) \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] dx$$

$$= 0 \text{ for } \forall g(x)$$

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0}$$

∴

Ex. Shortest path in 2D:

$$L = \int_{x_1}^{x_2} \sqrt{1+y'^2} dx$$

$$\text{so } f = f(y, y', x) = \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = - \frac{d}{dx} \left(\frac{1}{2} \frac{2y'}{\sqrt{1+y'^2}} \right) = 0$$

$$\text{so } \frac{y'}{\sqrt{1+y'^2}} = \text{constant} \quad \rightarrow \quad y'^2 = C^2 (1+y'^2)$$

$$y'^2 = \frac{C^2}{1-C^2} = M^2 \quad \text{still a constant}$$

$$\Rightarrow dy = M dx$$

$$y' = M$$

$$= dy/dx$$

$$y(x) = mx + b$$

Ex. Fermat's Principle:

$$\text{For } n(x, y) = n, \text{ then } f(y, y', x) = \frac{n}{c} \sqrt{1+y'^2},$$

so travel time t is shortest when light travels in straight lines.

We know for a particle that with $\vec{r}(t), \vec{v}(t)$, we can reconstruct the trajectory.
The Lagrangian \mathcal{L} captures this information;

$$\mathcal{L}(\vec{r}, \vec{v}, t)$$

If $\vec{r}(t)$ is the true path travelled by the particle from $t_1 \rightarrow t_2$, then the integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{r}, \dot{\vec{r}}, t) dt \quad [\text{Action}]$$

will be an extremum (stationary).

Hamilton's Principle (The Principle of Least Action):

The actual path travelled by a particle between t_1, t_2 is that which makes S stationary.

In 3D,

$$\mathcal{L}(x, \dot{x}, y, \dot{y}, z, \dot{z}, t)$$

so define

$$S(q, \dot{q}, t) = \int \mathcal{L}(x + q^f, \dot{x} + \dot{q}^f, y + q^g, \dot{y} + \dot{q}^g, z + q^h, \dot{z} + \dot{q}^h, t) dt$$

$$\text{repeat } \frac{dS}{dq} = \frac{dS}{d\dot{q}} = \frac{dS}{d\gamma} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

\Rightarrow

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{x}} = 0$$

Three DOF \Rightarrow Three EOM!

(6)



Physics 411 - Winter 2015

Wednesday, Week 7 ?

Review:

$$S(y, y', x) = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx$$

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0}$$

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{r}_1, \vec{v}_1, \dots, \vec{r}_N, \vec{v}_N, t) dt$$

$\left(\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array} \right)$ $\left(\begin{array}{c} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{array} \right)$

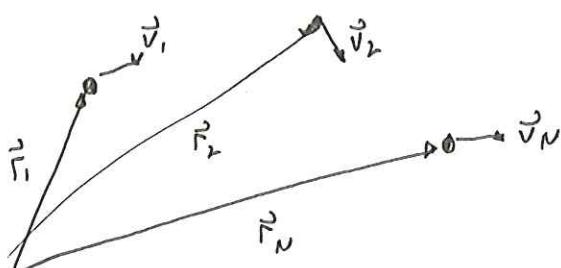
$$\boxed{\frac{\partial \mathcal{L}}{\partial \vec{r}_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \vec{v}_i} = 0}$$

Cartesian

Hamilton's Principle or the Principle of Least Action states path will minimize S .

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0}$$

Generalized



For N particles, we can have max of $3 \cdot N$ degrees of freedom.

The system can be described by "n" generalized coordinates, q_1, \dots, q_n .

If $n < 3 \cdot N$, then the system is constrained.

If $n = D.O.F.$, then the system is said to be Holonomic

In general,

$$\# \text{DOF} = \# \text{EOM}$$

If we have N particles, then we'll have N position vectors

$$\vec{r}_q \quad q = 1, \dots, N$$

in a maximum of $3N$ DOF. We can describe positions using generalized coordinates:

$$\vec{r}_q = \vec{r}_q(q_1, \dots, q_n, t)$$

of generalized coordinate

$$n \leq 3N \quad \text{Cartesian}$$

$$q_i = q_i(\vec{r}_1, \dots, \vec{r}_N, t) \quad i = 1, \dots, n \quad \text{Generalized}$$

In general, $n \neq \text{DOF}$, but we'll work with holonomic systems where $n = \text{DOF}$.

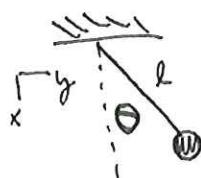
Then,

$$\{ (q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, t) \}$$

$$n < 3N \Rightarrow \text{constrained}$$

$$q_i(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t)$$

Ex]



$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l \cos \theta \\ l \sin \theta \end{pmatrix}$$

$$q_1 = \theta$$

$$n = 1$$

This system is constrained!

$\{ \}$ contains all the information, but can $\{ \}'$'s be different & still give same $\vec{r}(t)$ (matrix)?

Yes! $\{ \}'(q, \dot{q}, t) = \{ \}(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$ *total dt of a function of q, t only.*

$$\frac{d}{dq} \int \frac{d}{dt} (f(q + \alpha g(t), t)) dt = \int \frac{d}{dt} \left(\frac{\partial f}{\partial q} g(t) \right) dt = \left. \frac{\partial f}{\partial q} g(t) \right|_{t_1}^{t_2} = 0$$

\therefore so $\{ \}$ & $\{ \}'$ are the same under variation.

(2) ~~10~~
~~15~~

We already told you $\mathcal{L} = T - V$, but why? Let's infer \mathcal{L} for a free particle from the symmetry of space & time.

Form of \mathcal{L} for a free particle:

$$\text{we know } \mathcal{L} = \mathcal{L}(\vec{r}, \vec{v}, t)$$

The homogeneity of space & time gives:

↓
Translational
Invariance

No \vec{r} or $| \vec{r} |$ dependence

No time dependence

No t

$$\Rightarrow \mathcal{L}(\vec{v})$$

Isotropy of space demands \mathcal{L} be rotationally-invariant, therefore

$$\mathcal{L} \approx \mathcal{L}(\vec{v}) \Rightarrow \mathcal{L}(\vec{v}^2) \quad (\text{this includes } |\vec{v}| \text{-dependence})$$

Tangent: At this point, what does Euler-Lagrange Eq. tell us?

Also, the Euler-Lagrange Eq. gives:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \vec{v}} = \text{constant}$$

so looking at the \dot{x} -term in a Taylor series:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}} & \left(a_0 + a_1 \dot{x} + a_2 \dot{x}^2 + \dots \right) \\ &= 2a_1 \ddot{x} + 4a_2 \dddot{x}^3 + \dots \\ &= \text{constant} \end{aligned}$$

so $\ddot{x} = \text{constant}$

$$\boxed{\vec{v} = \text{constant}}$$

Principle of inertia!!

Follows from Homo, Iso of space & Homo of time + E-L eq.

Form of \mathcal{L} for free particle:

Now state $f(v)$ in S , $f(\vec{v} + \vec{\epsilon})$ in S' where $\vec{\epsilon}$ is velocity of S . Now

$$\mathcal{L}' = \mathcal{L}((\vec{v} + \vec{\epsilon})^2) = \mathcal{L}(v^2 + 2\vec{v} \cdot \vec{\epsilon} + \vec{\epsilon}^2)$$

$$= \mathcal{L}(v^2) \approx \mathcal{L}(v^2) + \frac{\partial \mathcal{L}}{\partial v^2} 2\vec{v} \cdot \vec{\epsilon}$$

$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial v^2} 2\vec{v} \cdot \vec{\epsilon}$$

$$f(x+dx) \approx f(x) + \frac{df}{dx} dx$$

$$\frac{d}{dt} f(\vec{r}, t) \text{ iff } \frac{\partial \mathcal{L}}{\partial v^2} = \text{constant}$$

$$\text{since } \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v^2} 2\vec{F} \cdot \vec{\epsilon} \right)$$

$$= \frac{\partial \mathcal{L}}{\partial v^2} 2\vec{v} \cdot \vec{\epsilon}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial v^2} = \text{constant}$$

Mass is the largest parameter available

$$\therefore \mathcal{L} = M c v^2 \quad \text{we call } C = \frac{1}{2} M$$

$$\boxed{\mathcal{L} = \frac{1}{2} M V^2}$$

For finite $\vec{v}' = \vec{v} + \vec{V}_0$ $L' = \frac{1}{2} M V'^2 = \frac{1}{2} M (\vec{v} + \vec{V}_0)^2$

$$= \frac{1}{2} M V^2 + M \vec{v} \cdot \vec{V}_0 + \frac{1}{2} M V_0^2$$

$$= \frac{1}{2} M V^2 + \frac{d}{dt} (M \vec{v} \cdot \vec{V}_0 + \frac{1}{2} M V_0^2 \epsilon)$$

Total Derivative of t ... can ignore.

M20:

$$S = \int \frac{1}{2} M V^2 dt$$

If $M \ll \infty$, then S will have no minimum if V^2 were large.

We include interparticle interactions by an interaction potential: $U(\vec{r}_1, \dots, \vec{r}_N) \approx$

$U(q_1, \dots, q_n) \approx$

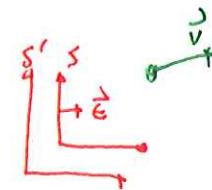
$$\begin{aligned} \mathcal{L} &= T - U \\ &= \sum \frac{1}{2} m_i \vec{v}_i^2 - U(\vec{r}_1, \dots, \vec{r}_N) \end{aligned}$$

infinitesimal.

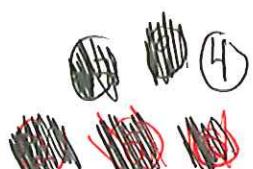
\vec{v}' is velocity of S' . Now

$$\vec{v}' = \vec{v} + \vec{\epsilon}$$

$$\vec{v} \in S$$



power of $\vec{\epsilon}$



Finally, Euler-Lagrange Equns are

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

For Holonomic systems, we have
 $i \in \{1, n\}$ where $n = \text{degree of freedom}$

Newton's Laws:

$$f = \frac{1}{2} m \vec{v}^2 - U(\vec{r}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{\partial U}{\partial x} - \frac{d}{dt} (m \dot{x})$$

$$= -\frac{\partial U}{\partial x} - m \ddot{x}$$

↑ If $\dot{x} = 0$

$$= -\frac{\partial U}{\partial x} - \dot{p}_x$$

Also one for $y \not\equiv z \neq 0$,

$$\begin{pmatrix} m \ddot{x} \\ m \ddot{y} \\ m \ddot{z} \end{pmatrix} = \begin{pmatrix} -\partial U / \partial x \\ -\partial U / \partial y \\ -\partial U / \partial z \end{pmatrix} \Rightarrow m \vec{a} = -\vec{\nabla} U(\vec{r})$$

N.B. $\frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{p}_x \quad \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$

Ex $\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r, \theta)$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m r^2 \dot{\theta} \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \theta} = -\frac{\partial U}{\partial \theta} \quad [\text{Torque}]$$

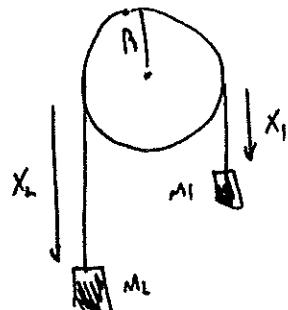
$$= Iw \quad [\text{Angular Momentum}]$$

(5)

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \underbrace{m r^2 \ddot{\theta}}_{Iw} + 2m r \dot{r} \dot{\theta} \quad \rightarrow \quad [\text{Torque}]$$

$\frac{\partial \mathcal{L}}{\partial q_i}$	Generalized Force
$\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$	Generalized Momentum

Ex]



$$x_1 + x_2 + \pi R = l$$

$$\dot{x}_1 = -\dot{x}_2$$

$$\begin{aligned} T &= \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 \\ &= \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_1^2 \\ &= \frac{1}{2} (M_1 + M_2) \dot{x}_1^2 \end{aligned}$$

$$\begin{aligned} U &= -M_1 g x_1 - M_2 g x_2 \\ &= -M_1 g x_1 - M_2 g (l - \pi R - x_1) \\ &= -(M_1 - M_2) g x_1 - M_2 g (l - \pi R) \end{aligned}$$

$$Q = \frac{1}{2} (M_1 + M_2) \dot{x}_1^2 + (M_1 - M_2) g x_1 + M_2 g (l - \pi R)$$

$$\begin{aligned} \frac{\partial L}{\partial x_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} &= (M_1 - M_2) g - \frac{d}{dt} ((M_1 + M_2) \dot{x}_1) \\ &= (M_1 - M_2) g - (M_1 + M_2) \ddot{x}_1 = 0 \end{aligned}$$

$$\Rightarrow \ddot{x}_1 = \frac{M_1 - M_2}{M_1 + M_2} g$$

$$\ddot{x}_1 = 0 \quad \text{when } M_1 = M_2$$

$$\ddot{x}_1 = g \quad M_1 \gg M_2$$

$$\ddot{x}_1 \leftarrow 0 \quad M_1 \ll M_2 \quad \checkmark$$

⑥

Physics 411 - Winter 2015

Friday, Week 7

Review: Uniqueness of \mathcal{L} :

$$\mathcal{L}' = \mathcal{L} + \frac{d}{dt} f(q, t)$$

for a free particle:

$$\begin{aligned} \mathcal{L} &= cv^2 \\ &= \frac{1}{2}mv^2 \end{aligned} \quad \stackrel{!}{\quad} \quad \vec{v} = \text{constant} \quad (\text{Principle of Inertia})$$

Hamilton's Principle $\int \mathcal{L} dt = \frac{1}{2}mv^2 \Rightarrow m > 0$

$$\boxed{\mathcal{L} = \sum_{i=1}^N \frac{1}{2}m_i \vec{v}_i^2 - U(\vec{r}_1, \dots, \vec{r}_N)}$$

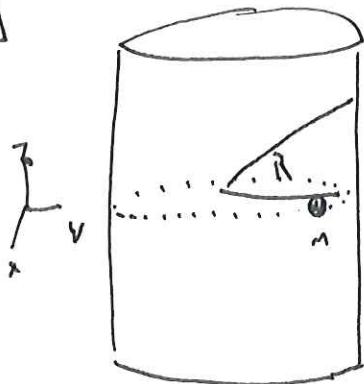
$$\frac{\partial \mathcal{L}}{\partial \vec{r}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = 0 \Rightarrow \vec{F} = m\vec{a}$$

$$\vec{r}_i(a_1, \dots, a_n)$$

$$\frac{\partial \mathcal{L}}{\partial a_i} = \text{Generalized Force}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{a}_i} = \text{Generalized Momentum}$$

Ex]



$$\vec{r} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ z \end{pmatrix}$$

$$\ddot{\vec{r}} = \begin{pmatrix} -R\dot{\phi}\sin\phi \\ -R\dot{\phi}\cos\phi \\ \ddot{z} \end{pmatrix}$$

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}m\ddot{z}^2$$

$$U = -mgz$$

$$L = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{1}{2}m\ddot{z}^2 - mgz$$

$$L(\dot{\phi}, \dot{z}, \ddot{z})$$

↓

No $\dot{\phi}$ -dependence

$$\frac{\partial L}{\partial \dot{\phi}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\phi}} = - \frac{d}{dt} (mR^2\dot{\phi}) = 0$$

$\Rightarrow \dot{\phi}$ is an ignorable coordinate ... $\dot{\phi}$

$$\frac{\partial L}{\partial \dot{\phi}} = \text{constant.}$$

\Rightarrow

$$\boxed{mR^2\dot{\phi} = \text{constant}} \\ = l$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} = -mg - \frac{d}{dt} (m\ddot{z})$$

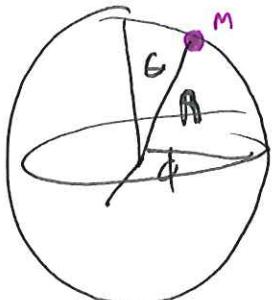
$$= -mg - m\ddot{z}$$

or

$$\boxed{m\ddot{z} = -mg}$$

Ex)

Spherical
pendulum



$$\vec{r} = \begin{pmatrix} R \sin \theta \cos \phi \\ R \sin \theta \sin \phi \\ R \cos \theta \end{pmatrix}$$

$$\ddot{\vec{r}} = \begin{pmatrix} \dot{\theta} R \cos \theta \cos \phi + \dot{\phi} R \sin \theta \sin \phi \\ \dot{\theta} R \cos \theta \sin \phi + \dot{\phi} R \sin \theta \cos \phi \\ -\dot{\theta} R \sin \theta \end{pmatrix}$$

$$T = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m \left[\dot{\theta}^2 R^2 \cos^2 \theta + \dot{\phi}^2 R^2 \sin^2 \theta + \right.$$

$\left. - 2 \dot{\theta} \dot{\phi} R^2 \cos \theta \sin \theta \cos \phi \sin \phi \right]$
 $+ \dot{\theta}^2 R^2 \cos^2 \theta \sin^2 \phi + \dot{\phi}^2 R^2 \sin^2 \theta \cos^2 \phi$
 $+ 2 \dot{\theta} \dot{\phi} R^2 \cos \theta \sin \theta \sin \phi \cos \phi$
 $\left. + \dot{\theta}^2 R^2 \sin^2 \theta \right]$

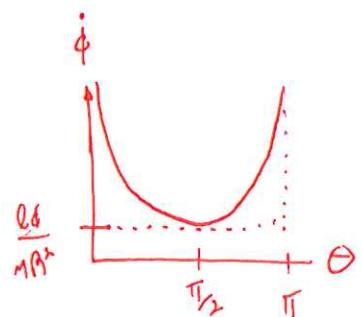
skip

$$= \frac{1}{2} m \left(\dot{\theta}^2 R^2 \cos^2 \theta + \dot{\phi}^2 R^2 \sin^2 \theta + \dot{\theta}^2 R^2 \sin^2 \theta \right)$$

$$= \boxed{\frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2 \theta \dot{\phi}^2}$$

$$U = +mgz$$

$$= +mgR \cos \theta$$



$$L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2 \theta \dot{\phi}^2 - mgR \cos \theta$$

$$\dot{\phi} = \frac{l_\phi}{m R^2 \sin^2 \theta}$$

$$= L(\theta, \dot{\theta}, \dot{\phi})$$

↳ Leads to a
conserved quantity!

$\dot{\phi}:$

$$\frac{dL}{d\dot{\phi}} - \frac{d}{dt} \frac{dL}{d\dot{\phi}} = - \frac{d}{dt} (m R^2 \sin^2 \theta \dot{\phi}) = 0$$

$$\boxed{m R^2 \sin^2 \theta \dot{\phi} = \text{constant}}$$

(3)

$$\Theta: \frac{d\ddot{\theta}}{d\theta} - \frac{d}{dt} \frac{d\theta}{d\theta} = mR^2 \sin\theta \cos\theta \dot{\phi}^2 + mgR \sin\theta - \frac{d}{dt} (mR^2 \dot{\theta})$$

or $mR^2 \ddot{\theta} = mR^2 \sin\theta \cos\theta \dot{\phi}^2 + mgR \sin\theta$

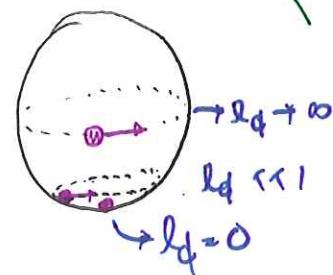
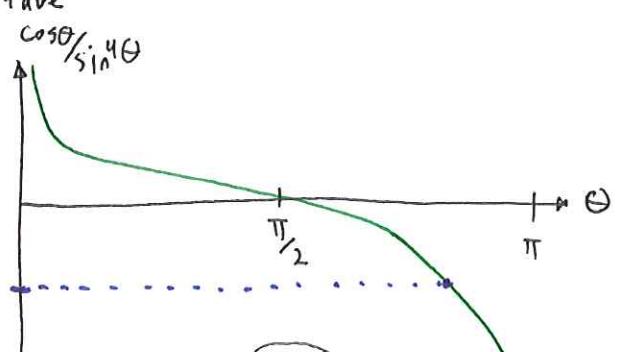
use $\dot{\phi} = \frac{l_\phi}{mR^2 \sin^2 \theta}$

$$mR^2 \ddot{\theta} = \frac{l_\phi^2}{mR^2} \cdot \frac{\cos\theta}{\sin^3\theta} + mgR \sin\theta$$

Very Non-linear

Equilibrium values: with $\ddot{\theta} = 0$, we have

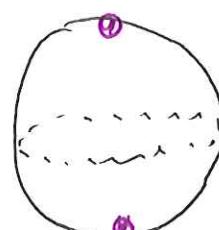
$$\frac{\cos\theta}{\sin^4\theta} = -\frac{gM^2R^3}{l_\phi^2}$$



The case of no angular momentum - $l_\phi = 0$: $\dot{\phi} = 0$ too!

$$\ddot{\theta} = \frac{g}{R} \sin\theta$$

w/ $\theta = \theta_0 + \epsilon$

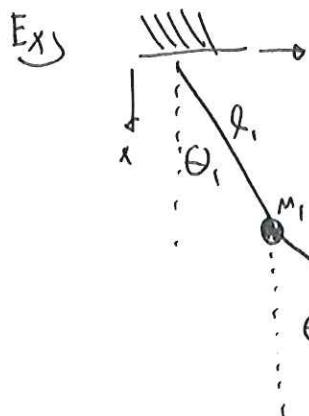


$$\ddot{\theta} = \ddot{\epsilon} = \frac{g}{R} \sin\theta_0 + \frac{g}{R} \cos\theta_0 \cdot \epsilon$$

$$= \begin{cases} \frac{g}{R} & \theta_0 = 0 \\ -\frac{g}{R} & \theta_0 = \pi \end{cases}$$

unstable

stable



$$\vec{r}_1 = l_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}$$

$$\vec{r}_2 = l_1 \begin{pmatrix} -\dot{\theta}_1 \sin \theta_1 \\ \dot{\theta}_1 \cos \theta_1 \end{pmatrix}$$

$$\vec{r}_r = \vec{r}_1 + l_2 \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}$$

$$\vec{r}_2 = \vec{r}_1 + l_2 \begin{pmatrix} -\dot{\theta}_2 \sin \theta_2 \\ \dot{\theta}_2 \cos \theta_2 \end{pmatrix}$$

$$T = \frac{1}{2} M_1 \dot{r}_1^2 + \frac{1}{2} M_2 \dot{r}_2^2$$

$$= \frac{1}{2} M_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} M_2 \left(\dot{r}_1^2 + l_2^2 \dot{\theta}_2^2 \right)$$

$$+ 2l_1 l_2 \left(+ \dot{\theta}_1 \dot{\theta}_2 \sin \theta_1 \sin \theta_2 \right. \\ \left. + \dot{\theta}_1 \dot{\theta}_2 \cos \theta_1 \cos \theta_2 \right)$$

skip

$$= \frac{1}{2} M_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} M_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 \right.$$

$$\left. + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

$$U(\vec{r}_1, \vec{r}_2) =$$

$$= -M_1 g X_1 - M_2 g X_2$$

$$= -M_1 g l_1 \cos \theta_1 - M_2 g l_2 \cos \theta_2 - M_2 g l_1 \cos \theta_1$$

skip:

$$L = \frac{1}{2} M_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} M_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

$$S(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$$

$$+ (M_1 + M_2) g l_1 \cos \theta_1 + M_2 g l_2 \cos \theta_2$$

$\dot{\theta}_1$:

$$\frac{dL}{d\dot{\theta}_1} = -\frac{1}{2} M_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (M_1 + M_2) g l_1 \sin \theta_1$$

$$- M_1 l_1^2 \ddot{\theta}_1 - M_2 l_2^2 \ddot{\theta}_2 - \frac{d}{dt} \left(2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

skip

$$- \frac{1}{2} M_2 \left(2l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + 2l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2) (\ddot{\theta}_1 - \ddot{\theta}_2) \right)$$

(5)

$$= - (m_1 + m_2) g l_1 \sin \theta_1$$

$$- (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$+ m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0$$

+ Another one... $m_1 = m_2 = m$ $l_1 = l_2 = l$ in $\theta_1 \ll 1, \theta_2 \ll 1$:

$$2l \ddot{\theta}_1 + l \ddot{\theta}_2 = -2g\theta_1$$

$$l \ddot{\theta}_1 + 2l \ddot{\theta}_2 = -g\theta_2$$

$$\begin{pmatrix} 2l & l \\ l & 2l \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -2g & 0 \\ 0 & -g \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$



Coupled oscillators

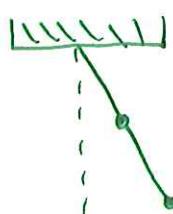
$$\mathbf{M} \ddot{\mathbf{X}} = -K \mathbf{X}$$

↓ ↓
Matrix Matrix

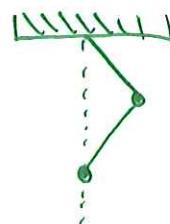
Talk about this last week
of class.

Already saw one example w/ $\vec{F} = q\vec{v} \times \vec{B}$.

Will have eigen modes:



In Phase



Out of Phase

Ex] Change in \vec{B} & \vec{B} field: $\boxed{\vec{F} = m\vec{a} \Rightarrow \vec{f}}$

$$\vec{g} = \frac{1}{2} M \vec{r}^2 - q(V - \vec{r} \cdot \vec{A}) \xrightarrow{\text{Electrical potential}} \text{Vector potential}$$

$$= \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q(V - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z)$$

$$\frac{d\vec{g}}{dt} - \frac{d}{dt} \frac{d\vec{g}}{dx} = -q \left(\frac{\partial V}{\partial x} - \dot{x} \frac{\partial A_x}{\partial x} - \dot{y} \frac{\partial A_x}{\partial y} - \dot{z} \frac{\partial A_x}{\partial z} \right)$$

$$\underbrace{- \frac{d}{dt} \left(\frac{\partial}{\partial t} \left(M\dot{x} + qA_x \right) \right)}_{\text{full time derivative}} \xrightarrow{\text{Generalized Momentum}}$$

$$= -M\ddot{x} \Rightarrow q \left(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} \right)$$

$$\Rightarrow M\ddot{x} = -q \left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t} \right) + q\dot{y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + q\dot{z} \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right)$$

$$= q \left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \right)_x + q \left(\vec{v} \times (\vec{\nabla} \times \vec{A}) \right)_x$$

$$= q \left(E_x + (\vec{v} \times \vec{B})_x \right)$$

$qV = \text{energy}$
 $q\vec{A} = \text{momentum}$

$$\Rightarrow \text{Generalized Momentum} = \boxed{\vec{p} = M\vec{v} + q\vec{A}}$$

$$\Rightarrow M\vec{v} = \vec{p} - q\vec{A} \xrightarrow{\text{QM}} \boxed{-i\hbar \vec{\nabla} - q\vec{A} \text{ is momentum operator}}$$

Example of $m\vec{v} = \vec{p} - e\vec{A}$ in Quantum:

$$\vec{v} = \frac{1}{m} (-i\hbar\vec{\nabla} - e\vec{A})$$

The Meissner effect states $B=0 \quad \& \quad \vec{J}=0$

In a superconductor we have the electron condensate :

$$\Psi = n^{1/2} e^{i\theta(\vec{r})}$$

Like photons

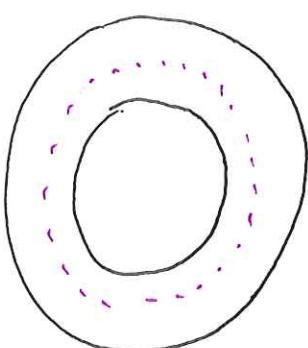
$$\Psi^* = n^{1/2} e^{-i\theta(\vec{r})}$$

$$\begin{aligned}\vec{J} &= qn\vec{v} \\ &= q\Psi^*\vec{\nabla}\Psi \\ &= \frac{q}{m}\Psi^* \left(-i\hbar\vec{\nabla}\Psi - e\vec{A}\Psi \right) \\ &= \frac{q}{m}\Psi^* \left(-i\hbar i\partial_r\Psi - e\vec{A}\Psi \right)\end{aligned}$$

$$= \frac{q}{m} \cdot n \left(\hbar\vec{\nabla}\theta(\vec{r}) - e\vec{A} \right)$$

$$= 0$$

$$\boxed{\hbar\vec{\nabla}\theta(\vec{r}) = e\vec{A}}$$



$$\oint \hbar\vec{\nabla}\theta \cdot d\vec{l} = e \oint \vec{A} \cdot d\vec{l}$$

$$\hbar(\theta_2 - \theta_1) = e \int \vec{\nabla} \times \vec{A} \cdot d\vec{a}$$

$$= e \int \vec{B} \cdot d\vec{a}$$

$$= e \Phi_B$$

$$\boxed{\Phi_B = \frac{\hbar\omega\theta}{q} = \frac{2\pi\hbar}{q} \cdot n \quad n \in \mathbb{Z}}$$

Flux
Quantization