

Physics 411 - Winter 2015

Monday, Week 8 :

Superconductor:

$$\mathcal{L} = \frac{1}{2} m \dot{\vec{r}}^2 - e(V(\vec{r}) - \dot{\vec{r}} \cdot \vec{A})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \vec{p} = m\vec{v} + e\vec{A}$$

$$\vec{v} = \frac{1}{m} (-i\hbar \vec{\nabla} - e\vec{A}) \quad \text{"Velocity Operator in Q.M."}$$

$$\psi = e^{i(kx - \omega t)}$$

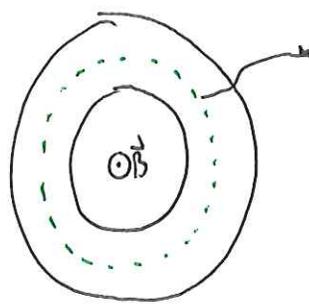
$$\begin{aligned} \langle p \rangle &\equiv \int \psi^* \hat{p} \psi dx = \int \psi^* -i\hbar \frac{d}{dx} \psi dx \\ &= \int -i\hbar \cdot ik \psi^* \psi dx \\ &= \hbar k \int |\psi|^2 dx \\ &= \hbar k \end{aligned}$$

$$\begin{aligned} \psi &= n^{1/2} e^{i\theta(\vec{r})} \\ |\psi| &= n \rightarrow \text{\# density of electrons} \end{aligned}$$

$$\begin{aligned} \vec{j} &= e \psi^* \vec{v} \psi \quad \text{[Current density]} \\ &= e \psi^* \frac{1}{m} (-i\hbar \vec{\nabla} - e\vec{A}) \psi \\ &= \frac{e}{m} \psi^* (\hbar \vec{\nabla} \theta(\vec{r}) - e\vec{A}) \psi \\ &= \frac{e}{m} \cdot n (\hbar \vec{\nabla} \theta(\vec{r}) - e\vec{A}) \\ &= 0 \end{aligned}$$

$$\therefore \hbar \vec{\nabla} \theta(\vec{r}) = e\vec{A}$$

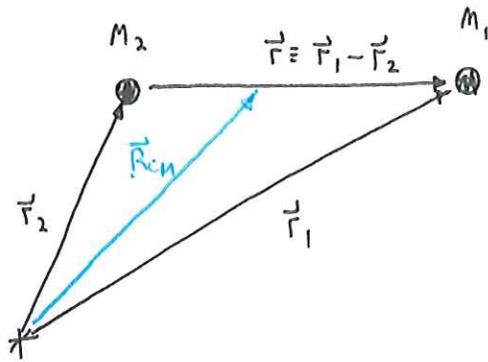
$$\Rightarrow \Phi_B = \frac{2\pi \cdot \hbar}{e} \cdot n \quad n \in \mathbb{Z}$$



$$\begin{aligned} \oint \hbar \vec{\nabla} \theta(\vec{r}) \cdot d\vec{e} &= \oint e\vec{A} \cdot d\vec{e} \\ &= e \oint \vec{\nabla} \times \vec{A} \cdot d\vec{a} \\ &= e \oint \vec{B} \cdot d\vec{a} \\ &= e \Phi_B \\ &= \hbar \cdot 2\pi \cdot n \end{aligned}$$

Quantized B-flux is at the heart of all Superconducting Quantum Devices & Circuits.

The Two-Body Central Force Problem:



$m_1 \dot{\epsilon}_1 m_2$ interact via

$$\vec{F}(\vec{r}_1, \vec{r}_2) = \vec{F}(\vec{r}_1 - \vec{r}_2) \\ = F(\vec{r}_1 - \vec{r}_2) \hat{r}$$

i.e. Central
+ Conservative



rotationally invariant

Approach: Write \mathcal{L} & $\dot{\epsilon}$, solve E-L eq.

• First write \vec{r}_1, \vec{r}_2 in terms of $\vec{r} \dot{\epsilon}; \vec{R}_{CM}$:

$$\begin{aligned} \vec{R}_{CM} &\equiv \frac{1}{M} m_1 \vec{r}_1 + \frac{1}{M} m_2 \vec{r}_2 \\ &= \frac{m_1}{M} \vec{r}_1 + \frac{M - m_1}{M} \vec{r}_2 \\ &= \frac{m_1}{M} (\vec{r}_1 - \vec{r}_2) + \vec{r}_2 \\ &= \frac{m_1}{M} \vec{r} + \vec{r}_2 \quad \Rightarrow \end{aligned}$$

In CM Frame:

$$\vec{r}_1 = \frac{m_2}{M} \vec{r}$$

$$\vec{r}_2 = -\frac{m_1}{M} \vec{r}$$

$$\begin{aligned} \vec{r}_1 &= \vec{R}_{CM} + \frac{m_2}{M} \vec{r} \\ \vec{r}_2 &= \vec{R}_{CM} - \frac{m_1}{M} \vec{r} \end{aligned}$$

• The Lagrangian:

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - U(\vec{r}_1, \vec{r}_2)$$

$$\hookrightarrow U(\vec{r}_1 - \vec{r}_2) = U(r)$$

$$= \frac{1}{2} m_1 \left(\dot{\vec{R}}_{CM}^2 + \frac{m_2^2}{M^2} \dot{\vec{r}}^2 + \frac{2m_2}{M} \dot{\vec{r}} \cdot \dot{\vec{R}}_{CM} \right) + \frac{1}{2} m_2 \left(\dot{\vec{R}}_{CM}^2 + \frac{m_1^2}{M^2} \dot{\vec{r}}^2 - \frac{2m_1}{M} \dot{\vec{r}} \cdot \dot{\vec{R}}_{CM} \right) - U(r)$$

$$= \frac{1}{2} \underbrace{(m_1 + m_2)}_M \dot{\vec{R}}_{CM}^2 + \frac{1}{2} \left(\frac{m_1 m_2^2 + m_2 m_1^2}{M^2} \right) \dot{\vec{r}}^2 - U(r)$$

$$\underbrace{\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}}_{=} = \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} = \frac{m_1 m_2}{m_1 + m_2} = \mu$$

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}_{cm}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$$

We see $\dot{\vec{R}}_{cm}$ is ignorable ξ .

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}_{cm}} = M \dot{\vec{R}}_{cm} = \text{constant}$$

Furthermore, boosting to CM frame (which we can do since $\mathcal{L}_{cm} \xi \mathcal{L}$ will be equivalent up to $\frac{d}{dt} f(\vec{r}, t)$)

$$\dot{\vec{R}}_{cm} = 0 \quad \xi$$

$$\mathcal{L} = \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r) \quad \xi$$

$$\begin{aligned} \vec{r}_1 &= \frac{m_2}{M} \vec{r} & \dot{\vec{r}}_1 &= \frac{m_2}{M} \dot{\vec{r}} \\ \vec{r}_2 &= -\frac{m_1}{M} \vec{r} & \dot{\vec{r}}_2 &= -\frac{m_1}{M} \dot{\vec{r}} \end{aligned}$$

We can simplify this further by looking at the angular momentum:

The system is closed ξ we know from rotational invariance that $\vec{L}_{tot} = \text{constant}$. Here it is explicitly:

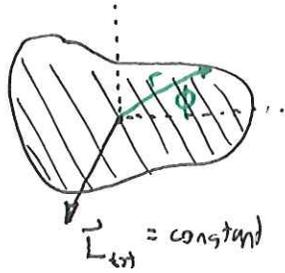
$$\begin{aligned} \vec{L}_{tot} &= m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2 \\ \dot{\vec{L}}_{tot} &= m_1 \dot{\vec{r}}_1 \times \dot{\vec{r}}_1 + m_1 \vec{r}_1 \times \ddot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 \times \dot{\vec{r}}_2 + m_2 \vec{r}_2 \times \ddot{\vec{r}}_2 \\ &= m_1 \vec{r}_1 \times \ddot{\vec{r}}_1 + m_2 \vec{r}_2 \times \ddot{\vec{r}}_2 \\ &= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} \\ &= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} \\ &= \vec{r} \times F(r) \hat{r} \\ &= 0 \end{aligned} \Rightarrow \boxed{\vec{L}_{tot} = \text{constant}}$$

Writing \vec{L}_{tot} in CM frame:

$$\begin{aligned} \vec{L}_{tot} &= m_1 \cdot \frac{m_2}{M} \vec{r} \times \dot{\vec{r}} + m_2 \cdot \frac{m_1}{M} \vec{r} \times \dot{\vec{r}} \\ &= M \vec{r} \times \dot{\vec{r}} \\ &= \boxed{\vec{r} \times \mu \dot{\vec{r}}} = \text{constant} \end{aligned}$$

What does this angular momentum relation tell us about the motion?

\vec{r}, \vec{v} in a fixed plane!



This 2D Planar motion lets us rewrite \mathcal{L} in polar coordinates:

$$\mathcal{L} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

$$= \mathcal{L}(r, \dot{r}, \dot{\phi}) \quad \text{Ignorable: } \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = l = \text{constant}$$

$$\dot{\phi} = \frac{l}{\mu r^2}$$

E:

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu r \dot{\phi}^2 - \frac{\partial U(r)}{\partial r} - \mu \ddot{r} = 0$$

Substitute

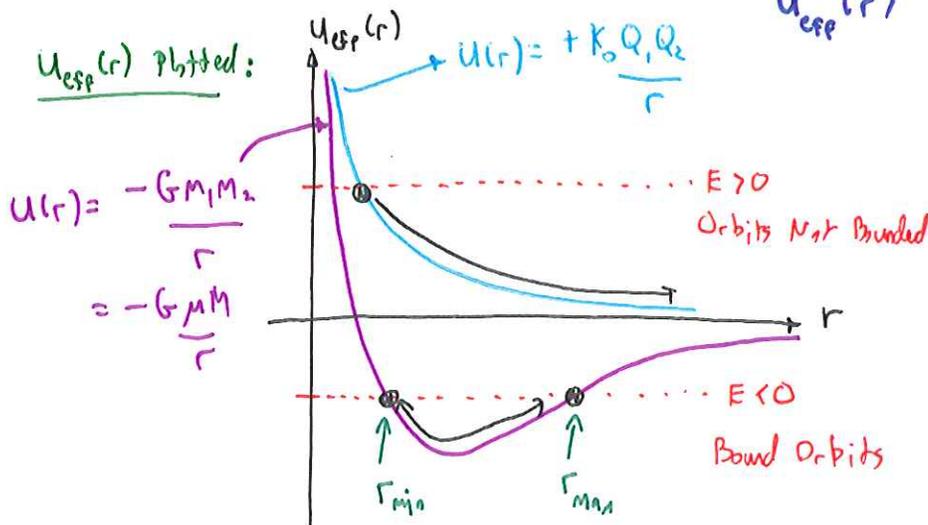
$$\mu \ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3} \quad \text{EOM}$$

$$\mu \ddot{r} = -\frac{\partial}{\partial r} \left[U(r) + \frac{l^2}{2\mu r^2} \right] = -\frac{\partial}{\partial r} U_{\text{eff}}(r)$$

N.B. The time-independence of \mathcal{L} guarantees:

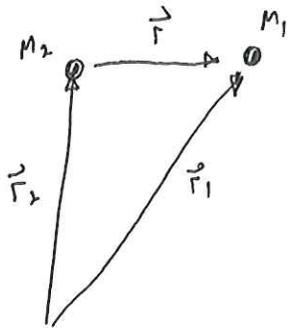
$$\mathcal{H} = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\phi}^2 + U(r) = E$$

Constant



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Tuesday, Week 8:



$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - U(\vec{r}_1, \vec{r}_2)$$

↓ CM frame

$$\mathcal{L} = \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$$

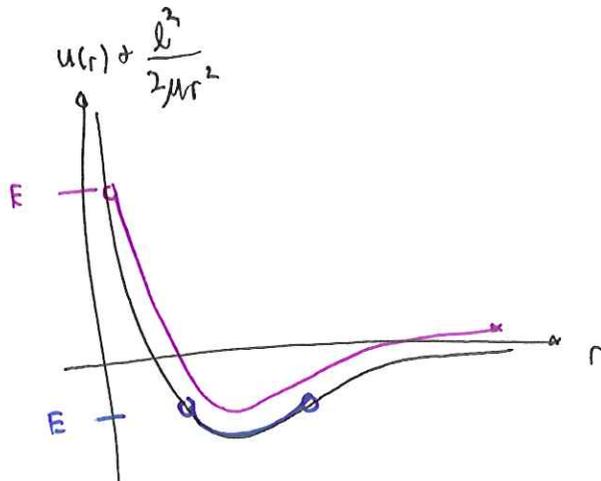
↓ ~~$\vec{r} \times \dot{\vec{r}} = \dot{\vec{\phi}}$~~ ~~$M \vec{r} \times \dot{\vec{r}} = \dot{\vec{L}}_{\text{tot}} = \text{constant}$~~

$$\mathcal{L} = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\phi}^2 - U(r)$$

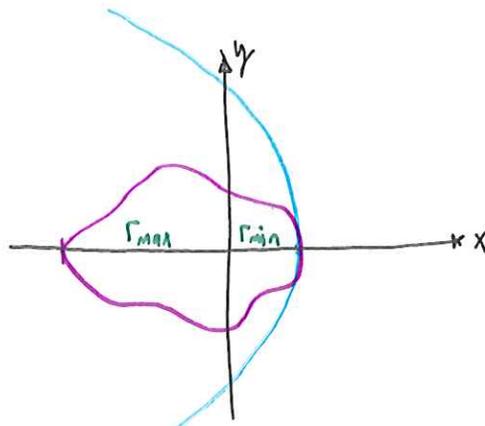
$$\downarrow \dot{\phi} = \frac{l}{\mu r^2}$$

$$\mu \ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{\mu r^3}$$

$$= -\frac{\partial}{\partial r} \left(U(r) + \frac{l^2}{2\mu r^2} \right)$$



$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\phi}^2 + U(r)$$



The position $\vec{r} = \vec{r}(\phi(t), r(t))$ is parametrized by time, but the solution to our problem is made simpler if we remove time by parametrizing w/ ϕ ; we need to ~~write~~ rewrite the EOM in terms of $d/d\phi$:

In general,

$$\frac{d}{dt} f(\phi(t)) = \frac{d\phi}{dt} \frac{df}{d\phi} = \dot{\phi} \frac{df}{d\phi} = \frac{l}{m r^2} \frac{\partial f}{\partial \phi}$$

Letting $r = 1/s$
 $s = 1/r$

$$\frac{df}{dt} = \frac{l}{m} s^2 \frac{df}{d\phi}$$

$$\frac{dr}{dt} = \frac{l}{m} s^2 \frac{\partial r}{\partial \phi} = \frac{l}{m} s^2 \frac{\partial}{\partial \phi} \left(\frac{1}{s} \right) = \left[-\frac{l}{m} \frac{\partial s}{\partial \phi} \right]$$

$$\frac{d^2 r}{dt^2} = \cancel{\frac{l}{m}} \frac{l}{m} s^2 \frac{d}{d\phi} \left[-\frac{l}{m} \frac{\partial s}{\partial \phi} \right] = - \left(\frac{ls}{m} \right)^2 \frac{d^2 s}{d\phi^2}$$

Then the EOM becomes,

$$m \ddot{r} = -\frac{2U}{r} + \frac{l^2}{m r^3} \longrightarrow -\frac{l^2 s^2}{m} \frac{d^2 s}{d\phi^2} = F(s) + \frac{l^2}{m} s^3$$

or

$$\frac{d^2 s}{d\phi^2} = -s - \frac{m}{l^2 s^2} F(s)$$

Now let's let $F(s)$ be the gravitational force:

$$F(r) = -\frac{G\mu M}{r^2} = -G\mu M \cdot s^2$$

so $F(s) = -\gamma S^2$

(ϵ)

$$S'' = -S + \frac{\mu\gamma}{l^2}$$

with $t(\phi) = s(\phi) - \gamma\mu/l^2$

$$t'' = -t$$

(ϵ)
 $t(\phi) = A \cos(\phi + \delta)$

$$s(\phi) = A \cos(\phi + \delta) + \gamma\mu/l^2$$

set to 0

$$= \frac{\gamma\mu}{l^2} \left(1 + A \frac{l^2}{\gamma\mu} \cos\phi \right)$$

$$\frac{1}{r} = \frac{1}{C} (1 + \epsilon \cos\phi)$$

$$\Rightarrow r(\phi) = \frac{C}{1 + \epsilon \cos\phi}$$

determined by initial conditions

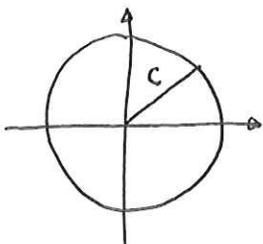
$$\epsilon \equiv \frac{A l^2}{\gamma\mu}$$

$$C \equiv \frac{l^2}{\gamma\mu}$$

What does $r(\phi)$ look like?

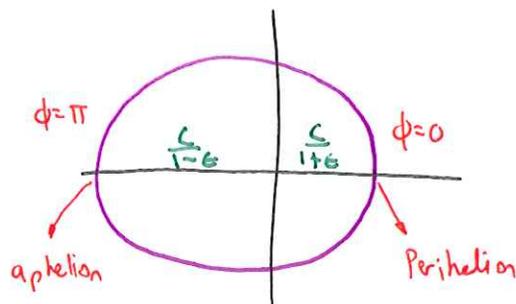
$\epsilon = 0$

$$r(\phi) = C$$



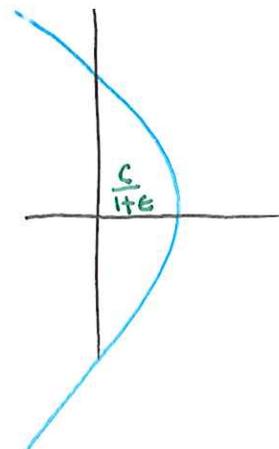
$\epsilon < 1$

Bounded Orbit



$\epsilon \geq 1$

Unbounded Orbit

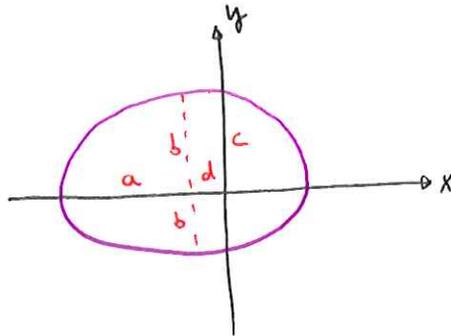


• Show Mathematics of Kepler Orbits

• The orbit can also be written (HW) as:

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Eq. for an ellipse

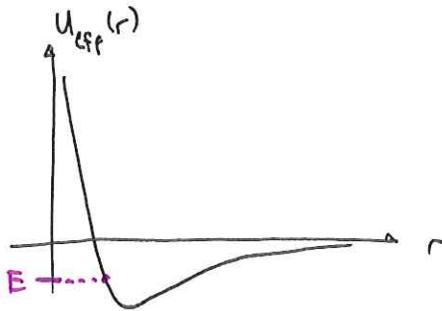


$a = \frac{c}{1-\epsilon^2}$ semimajor axis

$b = \frac{c}{\sqrt{1-\epsilon^2}}$ semiminor axis

$d = a\epsilon$

• E(ε):



$$\begin{aligned} E &= U_{\text{eff}}(r_{\text{min}}) \\ &= -\frac{\gamma}{r_{\text{min}}} + \frac{l^2}{2\mu r_{\text{min}}^2} \\ &= \frac{1}{2r_{\text{min}}} \left(\frac{l^2}{\mu r_{\text{min}}} - 2\gamma \right) \\ &= \frac{(1+\epsilon)}{2c} \left(\frac{l^2}{\mu c} (1+\epsilon) - 2\gamma \right) \\ &= \frac{(1+\epsilon)}{2c} (\gamma(1+\epsilon) - 2\gamma) \\ &= \frac{\gamma}{2c} (\epsilon^2 + 2\epsilon + 1 - 2 - 2\epsilon) \end{aligned}$$

$c = \frac{l^2}{\gamma\mu}$

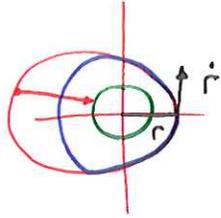
$$E(\epsilon) = \frac{\gamma}{2c} (\epsilon^2 - 1)$$

Thus, the ^{larger} ~~larger~~ E, the larger ε.

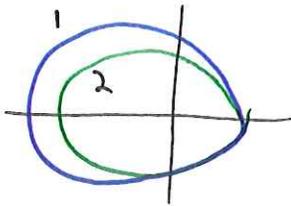
Changing the orbit:

Q? How do we change the orbit of a satellite? Say we have $m_{sat} \ll m_{earth}$, then $\vec{r}_2 = 0$.

∴ $\vec{r}_1 = \vec{r}$ ∴ we want



Say



$$\frac{c_1}{1 + \epsilon_1} = \frac{c_2}{1 + \epsilon_2}$$

$$c = \frac{l^2}{\gamma \mu}$$

$$l_1 = \mu r \dot{r}_1$$

$$l_2 = \mu r_2 \dot{r}_2$$

$$= \mu r_2 \lambda \dot{r}_1$$

$$= \lambda l_1$$

Momentum boost

$\lambda > 1$ forward thrust

$\lambda < 1$ backward thrust

Then,

$$\frac{l_1^2}{\gamma \mu (1 + \epsilon_1)} = \frac{\lambda^2 l_1^2}{\gamma \mu (1 + \epsilon_2)}$$

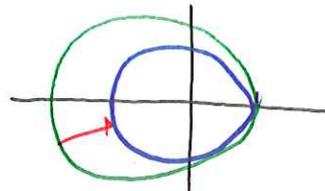
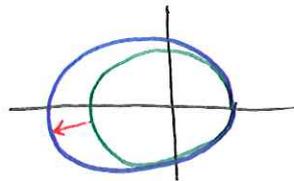
$$\Rightarrow \lambda^2 (1 + \epsilon_1) = 1 + \epsilon_2$$

$$\boxed{\epsilon_2 = \lambda^2 \epsilon_1 + (\lambda^2 - 1)}$$

$$\lambda > 1 \quad \epsilon_2 > \epsilon_1$$

$$\lambda = 1 \quad \epsilon_2 = \epsilon_1$$

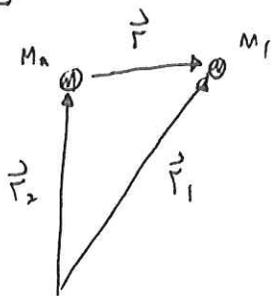
$$\lambda < 1 \quad \epsilon_2 < \epsilon_1$$



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Wednesday, Week 8:

Reviews:



$$M\ddot{r} = -\frac{\partial U}{\partial r} + \frac{l^2}{Mr^3}$$

$$\downarrow \dot{r} = -\frac{l}{m} \frac{ds}{d\phi} \quad s \equiv 1/r$$

$$\frac{ds}{d\phi^2} = -s - \frac{m}{l^2} F(s)$$

$$\downarrow F(s) = -\gamma s^2 \rightarrow F(r) = -\frac{\gamma}{r^2}$$

$$s'' = -s + \frac{M\gamma}{l^2}$$

$$s(\phi) = \frac{1}{c} (1 + \epsilon \cos \phi)$$

$$c \equiv \frac{l^2}{\gamma M} \quad \epsilon \equiv A \cdot c$$

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}$$

Show Mathematics.

Q: • ~~When~~ ^{translational} What is the Total Momentum of Two Body system in this ~~simulation~~ simulation?

• When will the angular velocity be largest? tangential velocity

$$\dot{\phi} = \frac{l}{Mr^2}$$

When r is smaller

$$v_t = r\dot{\phi} = \frac{l}{Mr}$$

$$\lambda < 1$$

$$E(\epsilon) = \frac{\gamma}{2c} (\epsilon^2 - 1)$$

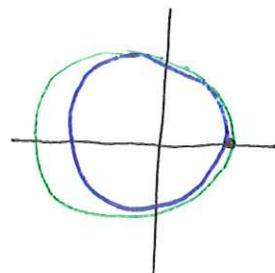
Changing orbits:

$$\epsilon_2 = \lambda^2 \epsilon_1 + (\lambda^2 - 1)$$

$$\frac{c_1}{1 + \epsilon_1} = \frac{c_2}{1 + \epsilon_2} \Rightarrow$$

$\lambda > 1$ forward burst

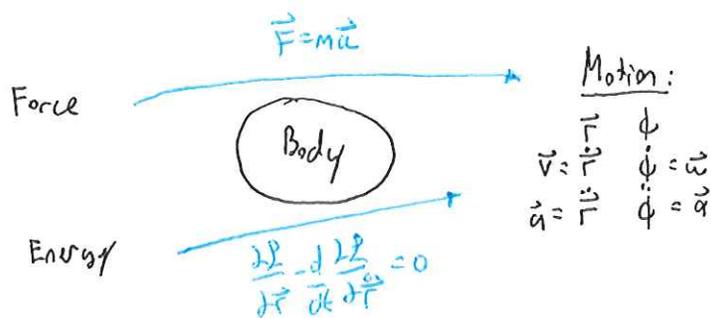
$\lambda < 1$ backward burst



$$c_2 = \lambda^2 c_1 \quad l_2 = \lambda l_1$$

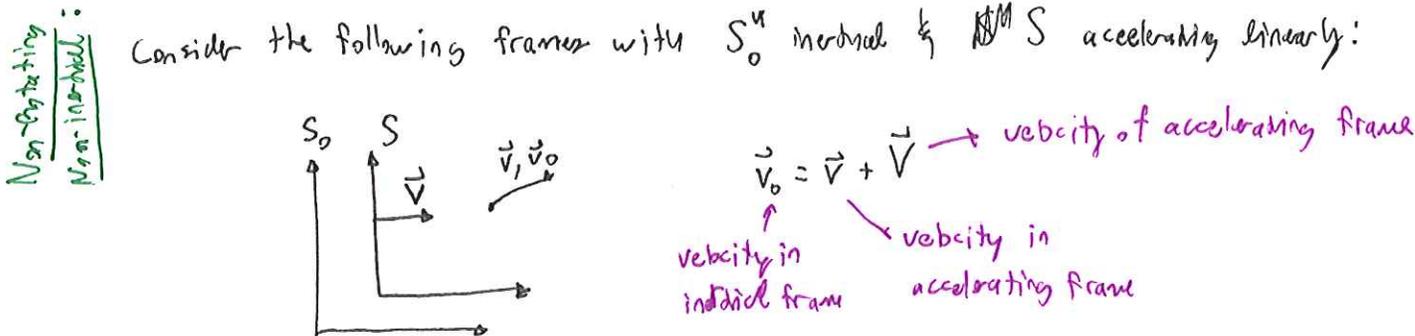
(1)

Where we are:



$\vec{F} = m\vec{a}$ holds in inertial reference frames. How can we rewrite N.2 in non-inertial frames?

Consider the following frames with S_0 inertial & S accelerating linearly:



$$\dot{\vec{r}}_0 = \dot{\vec{r}} + \dot{\vec{R}}$$

$\frac{d}{dt}$

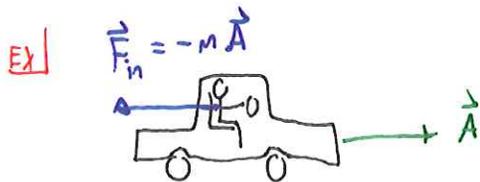
$$\ddot{\vec{r}}_0 = \ddot{\vec{r}} + \ddot{\vec{R}}$$

$$\Rightarrow \boxed{\vec{a} = \vec{a}_0 - \vec{A}}$$

$$\vec{a}_0 = \vec{a} + \vec{A}$$

$$m\vec{a} = \vec{F} + \vec{F}_{inertial}$$

Forces due to acceleration of frame: $-m\vec{A}$

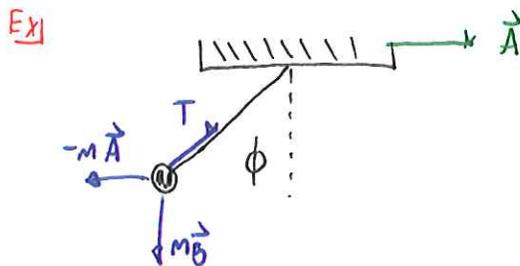


$$m\vec{a} = 0$$

$$= \vec{F} - m\vec{A}$$

$$\vec{F} = m\vec{A} \quad \checkmark$$

observed in inertial frame



observed in inertial

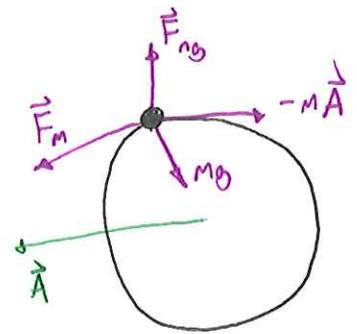
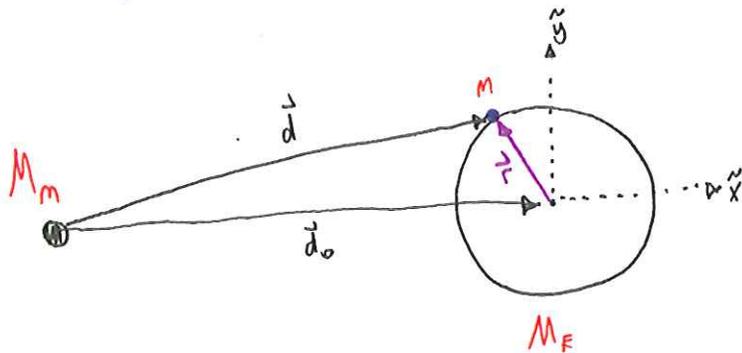
$$m\vec{a} = 0$$

$$= \vec{T} - m\vec{g} - m\vec{A}$$

$$\Rightarrow \vec{T} = m\vec{g} + m\vec{A} \quad \checkmark$$

Oceanic Tides :

- Describe the characteristics of tides?
- What do you think causes the tides?



$$\vec{A} = -\frac{GM_m}{d_0^2} \hat{d}_0$$

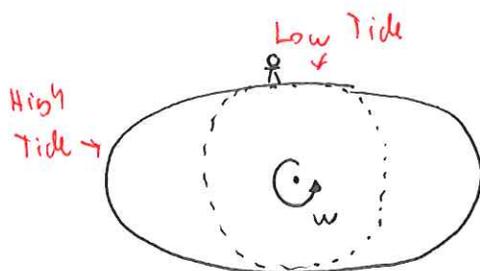
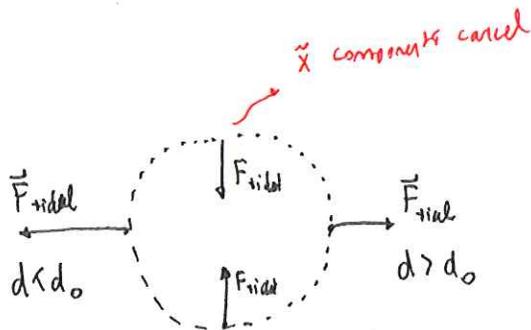
$$m\vec{r}'' = \vec{F} - m\vec{A}$$

$$m\vec{r}'' = m\vec{g} + \vec{F}_{ng} + \vec{F}_m - m\vec{A}$$

$$= m\vec{g} + \vec{F}_{ng} - \frac{GM_m \cdot m}{d^2} \hat{d} + \frac{GM_m \cdot m}{d_0^2} \hat{d}_0$$

$$= m\vec{g} + \vec{F}_{ng} - \underbrace{GM_m \cdot m \left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right)}_{\vec{F}_{tidal}}$$

Q: Which direction ~~where~~ will Tidal Force point on different parts of Earth?



Two Tides per day!!

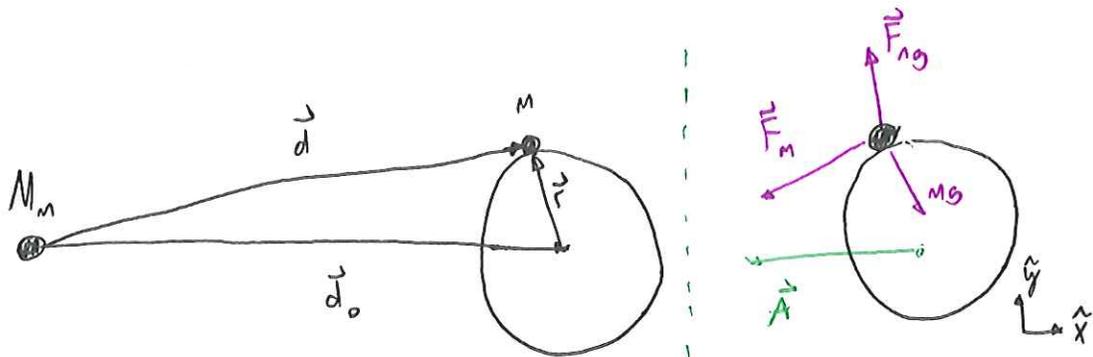
Physics 411 - Winter 2015

Friday, Week 8:

"N.2" in a linearly accelerated non-inertial frame:

$$m\vec{a} = \vec{F} - m\vec{A}$$

Oceanic Tides:



$$m\vec{a} = \vec{F} - m\vec{A}$$

$$= \vec{F}_{Ng} + m\vec{g} - \frac{GM_m m}{d^2} \hat{d} - m \left(-\frac{GM_m}{d_0^2} \hat{d}_0 \right)$$

$$= \vec{F}_{Ng} + m\vec{g} - \underbrace{\frac{GM_m m}{d^2} \left(\hat{d} - \frac{d_0}{d} \hat{d}_0 \right)}_{\vec{F}_{tidal}}$$

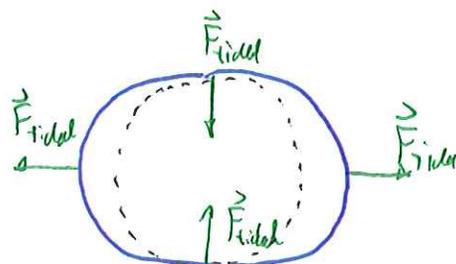
$$\underbrace{\hspace{10em}}_{\vec{F}_{tidal}}$$

conservative & \perp to surface

$$\Rightarrow \vec{F} = -\vec{\nabla}U$$

$$U = mgh - GM_m \cdot m \left(\frac{1}{d} + \frac{x}{d_0^2} \right)$$

$$\begin{aligned} \vec{A} &= \frac{1}{M_e} \left(-\frac{GM_e M_m}{d_0^2} \hat{d}_0 \right) \\ &= -\frac{GM_m}{d_0^2} \hat{d}_0 \end{aligned}$$



$$- \int_{-R_e}^{R_e} \left(GM_m m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial d_0^2} \right) \right) \cdot d\vec{r}$$

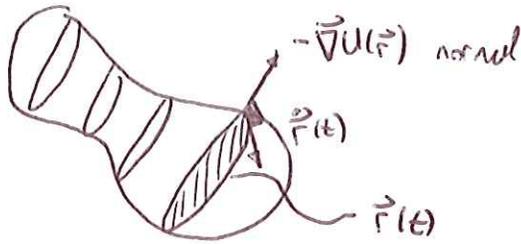
just $-\frac{GM_m m}{d}$

picks out \hat{x} comp

Q: What can we say about $U(\vec{r})$ if $\vec{F} = -\vec{\nabla}U$ is \perp to surface?

• $\vec{F} = -\vec{\nabla}U$ normal to surface $\Rightarrow U(\vec{r})$ is equipotential

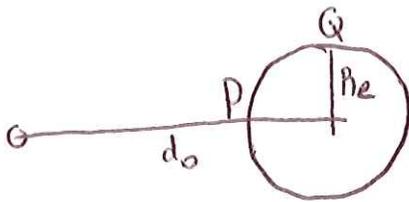
Proof:



$$\vec{\nabla}U \cdot \vec{r} = \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt}$$

$$= \frac{d}{dt} U(\vec{r}) = 0 \quad \forall \vec{r}(t)$$

$\Rightarrow U(\vec{r})$ is equipotential on surface.



$$U(Q) = U(P)$$

$$\approx mgh_Q - GM_M M \left(\frac{1}{d_Q} + \frac{x_Q}{d_0^2} \right)$$

$$= mgh_P - GM_M M \left(\frac{1}{d_P} + \frac{x_P}{d_0^2} \right)$$

$$mgh = -GM_M M \left(\frac{1}{\sqrt{d_0^2 + R_e^2}} \right) + GM_M M \left(\frac{1}{d_0 - R_e} - \frac{R_e}{d_0^2} \right)$$

$$\approx -GM_M M \frac{1}{d_0} \left(1 - \frac{R_e^2}{2d_0^2} \right) + GM_M M \left(\frac{1}{d_0} \left(1 + \frac{R_e}{d_0} + \frac{R_e^2}{d_0^2} \right) - \frac{R_e}{d_0^2} \right)$$

$$= \frac{3GM_M M R_e^2}{2d_0^3}$$

$$7.35 \times 10^{22} \text{ kg} \quad 6.37 \times 10^6 \text{ m}$$

$$\uparrow \quad \uparrow$$

$$3M_M R_e^4$$

$$2M_e d_0^3 \rightarrow 3.84 \times 10^8 \text{ m}$$

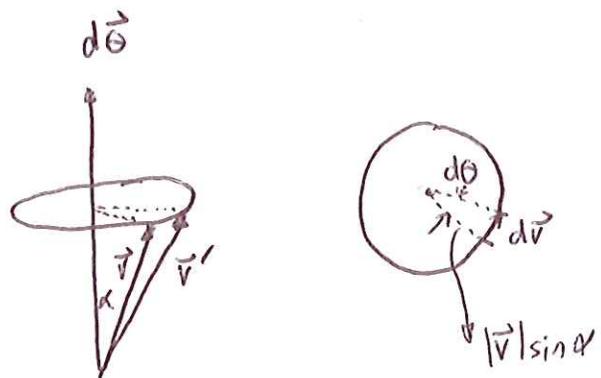
$$= 54 \text{ cm}$$

$$\Rightarrow \Delta h = \frac{3GM_M M R_e^2}{2mgh d_0^3} =$$

$$5.98 \times 10^{24} \text{ kg}$$

(2)

Rotating Reference Frame & Any vector \vec{v}



$$|d\vec{v}| = d\theta |\vec{v}| \sin \phi$$

$$d\vec{v} = d\vec{\theta} \times \vec{v}$$

Skip

$$d\vec{\theta}_1 + d\vec{\theta}_2 \longrightarrow (\vec{v} + d\vec{\theta}_1 \times \vec{v}) + d\vec{\theta}_2 \times (\vec{v} + d\vec{\theta}_1 \times \vec{v}) = \vec{v}_{final}$$

$$\downarrow$$

$$\vec{v} + d\vec{v} = \vec{v} + d\vec{\theta}_1 \times \vec{v} + d\vec{\theta}_2 \times \vec{v} + d\vec{\theta}_2 \times (d\vec{\theta}_1 \times \vec{v})$$

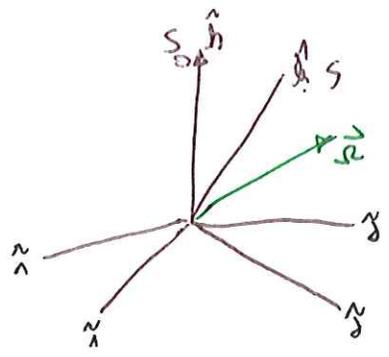
$\therefore d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$
 $\therefore d\vec{\theta}$ is a vector.

$$\frac{d\vec{v}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{v} = \vec{\omega} \times \vec{v}$$

ω is angular frequency of rotation & direction axis.

$\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}$ for any vector \vec{v} , even $\hat{i}, \hat{j}, \hat{k}$ constant

Consider \vec{v} in a rotating frame S which rotates with $\vec{\omega}$ relative to S_0 :



$\vec{v} = \hat{i} v_x + \hat{j} v_y + \hat{k} v_z$ so $\hat{i}(t)$ is S_0 frame.

$$\left(\frac{d\vec{v}}{dt}\right)_{S_0} = \frac{d\hat{i}}{dt} v_x + \hat{i} \frac{dv_x}{dt} + \frac{d\hat{j}}{dt} v_y + \hat{j} \frac{dv_y}{dt} + \frac{d\hat{k}}{dt} v_z + \hat{k} \frac{dv_z}{dt}$$

$$\left(\frac{d\vec{v}}{dt}\right)_{S_0} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k} + \frac{d\hat{i}}{dt} v_x + \frac{d\hat{j}}{dt} v_y + \frac{d\hat{k}}{dt} v_z$$

$$= \left(\frac{d\vec{v}}{dt}\right)_S + \vec{\omega} \times \vec{v}$$

Time derivative of \vec{v} in rotating frame.

Describe Terms

$$\frac{d\vec{v}}{dt} = \frac{S\vec{v}}{S t} + \vec{\omega} \times \vec{v}$$

Newton's 2nd Law in Rotating Frame:

$$\frac{d\vec{r}}{dt} = \frac{S\vec{r}}{S t} + \vec{\omega} \times \vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{S}{S t} \left(\frac{d\vec{r}}{dt}\right) + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \frac{S}{S t} \left(\frac{S\vec{r}}{S t} + \vec{\omega} \times \vec{r}\right) + \vec{\omega} \times \left(\frac{S\vec{r}}{S t} + \vec{\omega} \times \vec{r}\right)$$

$$= \frac{S^2\vec{r}}{S t^2} + \frac{S\vec{\omega}}{S t} \times \vec{r} + \vec{\omega} \times \frac{S\vec{r}}{S t} + \vec{\omega} \times \frac{S\vec{r}}{S t} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

⚠ (would be zero if S has rotational acceleration.)

$$= \frac{S^2\vec{r}}{S t^2} + 2\vec{\omega} \times \frac{S\vec{r}}{S t} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\text{Let } m \frac{d^2\vec{r}}{dt^2} = \vec{F} \quad \frac{S\vec{r}}{S t} = \dot{\vec{r}} \quad \frac{S^2\vec{r}}{S t^2} = \ddot{\vec{r}}$$

$$\Rightarrow m \ddot{\vec{r}} = \vec{F} + \underbrace{2m \dot{\vec{r}} \times \vec{\omega}}_{\text{Coriolis}} + m \underbrace{(\vec{\omega} \times \vec{r}) \times \vec{\omega}}_{\text{Centrifugal}}$$

(4)

Ex. $\vec{v} = \text{constant}$ in S_0 frame

$$\frac{d\vec{v}}{dt} = 0 \quad ; \quad \frac{S\vec{v}}{S t} = -\vec{\omega} \times \vec{v}$$

since S will be rotating w/ $-\vec{\omega}$

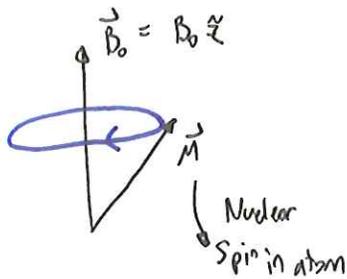
$\vec{v} = \text{constant}$ in S frame

$$\frac{S\vec{v}}{S t} = 0 \quad ; \quad \frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}$$

like normal

Example - Nuclear Magnetic Resonance: (The utility of the rotating frame)

constant \vec{B}_0 :



Q: Torque?

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{M} \times \vec{B}_0$$

$$\gamma \vec{L} = \vec{M}$$

γ = gyromagnetic ratio

$$\begin{aligned} \frac{d\vec{M}}{dt} &= \gamma \vec{M} \times \vec{B}_0 \\ &= \frac{d\vec{M}}{dt} + \vec{\omega} \times \vec{M} \end{aligned}$$

$$\begin{aligned} \text{or } \frac{d\vec{M}}{dt} &= \gamma \vec{M} \times \vec{B}_0 - \vec{\omega} \times \vec{M} \\ &= \vec{M} \times (\gamma \vec{B}_0 + \vec{\omega}) \end{aligned}$$

Larmor Frequency

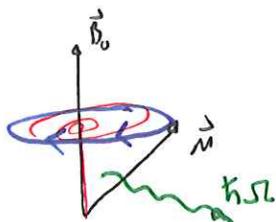
choose $\vec{\omega} = -\gamma \vec{B}_0$

\therefore Precession about \vec{z} axis

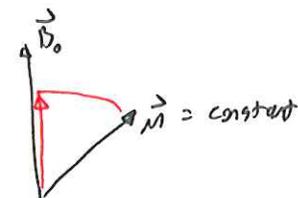
Q: Will \vec{M} rotate forever?

- Radiation
- of the bases in material

Rest Frame:



Rotating Frame:



The spins in real materials (numbering is $> 10^{23}$) will be randomly oriented before \vec{B}_0 is applied & will have magnetization M_0 in equilibrium after:

Q: What is magnetization of sample in equilibrium?

$$M_0 = n \cdot N \tanh\left(\frac{\mu B}{k_B T}\right)$$

$$\downarrow \uparrow \downarrow \leftarrow \rightarrow \Rightarrow \uparrow \uparrow \uparrow \uparrow \downarrow$$

To get big signal, NMR must take polarized spins from \uparrow to \rightarrow :

